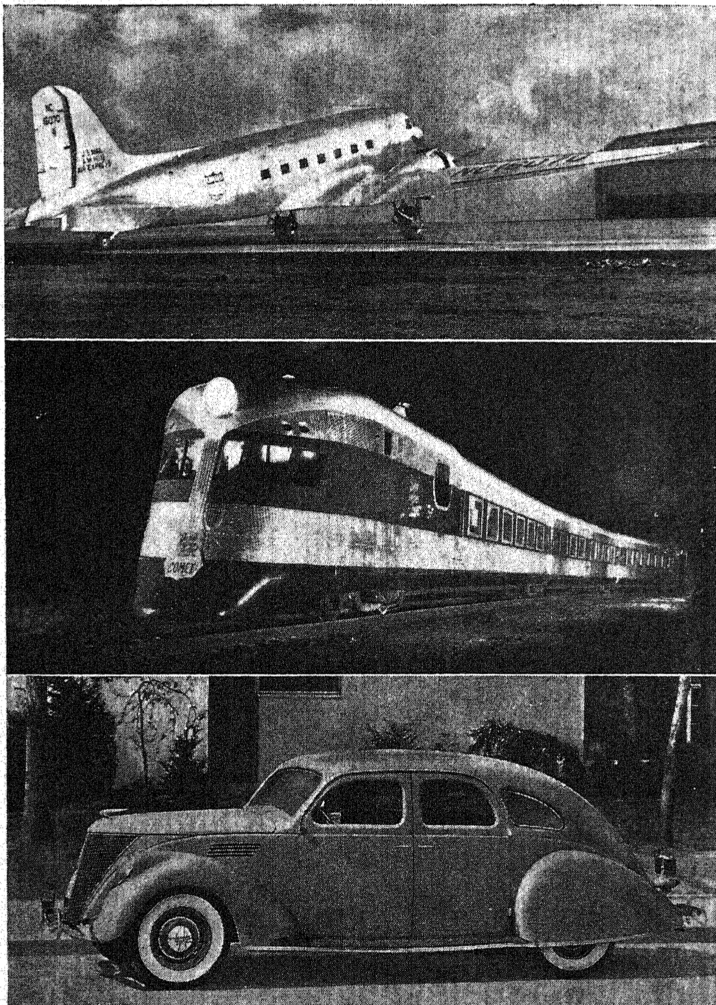

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STREAMLINE

Geometric curves and straight lines are combined in the design of modern airplanes, trains, and automobiles.

NEW
PLANE GEOMETRY

BY

JOHN C. STONE

AND

VIRGIL S. MALLORY

STATE TEACHERS COLLEGE, MONTCLAIR, NEW JERSEY

I think geometry the finest training the human mind can have. The whole art of reasoning is contained in its precepts. — PASCAL.

BENJ. H. SANBORN & CO.

CHICAGO 1938 BOSTON

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Printed in the United States of America

PREFACE

The widespread criticism of high school mathematics by those who would greatly alter the courses offered, or even eliminate them from the curriculum, has placed mathematics in a defensive position in many schools. Geometry has been subject to particular criticism. The authors, believing that the subject can retain its proper position in the secondary schools only if teaching materials and methods are constantly improved and perfected, offer this volume with the fervent hope that it may establish a new high standard among textbooks in plane geometry.

Free use has been made of materials appearing in the authors' *Modern Plane Geometry*. The wide use of that book and the willing co-operation of many teachers made possible an accurate evaluation of its content and method. The best has been retained.

Extensive experimentation by the authors in the classrooms at the College High School, Montclair, New Jersey, suggested new approaches to some topics and more effective methods for developing others. The nature of pupil-interests was carefully studied, and changing social emphases have been noted. The resulting book is modern in method, content, and attitude. Attention is called to certain specific features as follows:

Introduction.—The authors realize that the study of demonstrative geometry differs so much from the mathematics that the student has previously studied that

especial care must be used in the introduction to answer three questions for him. These are: "What is demonstrative geometry?" "Why is its study valuable?" and "How should it be studied?"

These questions are answered in several ways:

1. Through the history of the early uses of geometry in mensuration he is shown the contributions that the study of geometry has made to the development of civilization.
2. Numerous examples of the applications of geometry in art, industry, engineering, navigation, and astronomy are shown in illustrations and problems.
3. Unusual illustrations show the existence of geometric designs in the bodily structure of microscopic plant life, in animals, in coral, in sea shells, and in flowers.
4. Intuitive geometry and constructions serve as the vehicle for introducing fundamental definitions.
5. The student is led to see that conclusions reached through intuition and experimentation are liable to error and must be held tentatively.
6. He is shown the need and value of proof through logical reasoning to verify tentative conclusions.
7. Exercises in supplying reasons for statements give practice in use of the axioms and postulates.
8. Graded exercises dependent on the first two congruence theorems lead the student from simple one-step proofs through exercises gradually increasing in difficulty up to the use of overlapping triangles.

Psychological Development. — Plane geometry should develop the ability to understand deductive proofs and to

give them formally. Throughout the book the authors have borne in mind the additional fact that the course should also develop the student's powers of observation, should lead him to the formulation of discoveries about the relations in a figure, and finally, should direct him to the verification of these hypotheses by a proof. Moreover, originals should be so well graded and should lead up to the theorems in such a fashion that each one is met as an original, not more difficult than those originals that have previously been solved. Hence:

1. Each new unit of work is based on concepts and knowledge which are familiar to the student.
2. Only one difficulty is introduced at a time and adequate practice is given to make the student thoroughly familiar with each new development.
3. The student is led to make discoveries for himself. Statements of these are tentatively made, subject to verification by proof. Thus, observations and critical analysis make the subject vital and living.

Everyday Reasoning. — One of the features of the book is the use of many applications of logical reasoning to everyday decisions and discussions. Throughout the text the student's attention is directed to the valid inferences that may be drawn from given hypotheses and to the danger of forming conclusions which are not valid, or of reasoning in a vicious circle.

Flexibility. — Aware of the varying abilities of students, the authors have made it possible for the teacher to fit the course to their abilities in several ways. The student preparing for college entrance examinations will find that all of the requirements of the College Entrance Ex-

aminations Board and of the New York State Regents, and the suggestions of the National Committee on the Reorganization of Mathematics have been fully met in this book. Other differentiated courses are possible:

1. *Through selection of theorems.* — A minimum course is provided by proving formally only those theorems not marked with A or B. For the more capable students theorems marked B may be assigned. The A theorems provide honor work for the better students, but may be omitted entirely without destroying the sequence or interfering with the exercises.
2. *Through selection of exercises.* — Exercises down to the horizontal line form a minimum course. They are simple applications of the principles developed. Many of them are numerical, the solution of which depends only on simple applications without difficult work in algebra. Exercises below the line are more difficult, while starred exercises will challenge the best abilities of the brighter students.
3. *Through emphasis on practical applications.* — Other differentiation can be provided by stressing the numerical exercises and the practical applications of geometry.

Keyed Review and Testing Program. — At the end of each unit searching questions and complete summaries provide a review of the work of that unit. These are keyed so that the student can easily find the section where the work was first explained. Following this review there is a set of practice tests, also keyed, which adequately cover the work of the unit just completed. These are followed, in all units except the first, by a set

of keyed tests covering all of the work previously studied. It is realized that tests printed in the textbook are not best for determining the achievement of the students. Hence, a book of tests, containing objective tests on each unit and spiral cumulative tests, is provided for use with the text.

Theorems. — The number of formal theorems has been reduced to a minimum. Most of these occur as easy originals to be proved by the student before they are stated formally.

Exercises. — The authors' long classroom experience has convinced them that geometry should not be a feat of memory with exaggerated stress on formal theorems but that it should develop ability in discovering geometric relations and power in proving original exercises, and provide ample practice with numerical applications. To this end, graded original exercises, with the degree of difficulty increasing very gradually, lead the student to discover important geometric relations for himself and to anticipate and prove the theorems before they occur in the text. Following the theorems, wherever possible, many numerical applications of the principle are provided. The authors believe that this text contains more original exercises and more numerical exercises than any other.

Historical Notes. — It is the authors' conviction that historical allusions and anecdotes should be introduced at appropriate points throughout the course instead of being developed at a single point. Such material has been placed throughout the text, wherever its use tends to illustrate a principle or to add interest to the work.

Practical Applications. — Over two hundred exercises show the practical applications of geometry. While it is

realized that such applications are not a major objective in the teaching of plane geometry in high schools, their use provides interest and motivation.

Locus. — The difficulty which students have with this topic led the authors to experiment in their high school classes. As a result of this experimentation a new development is given in this book. *First,* the attention of the student is called to the many illustrations of a point moving under various restrictions and to the resulting figures that are formed. Drawings of linkages aid in showing this. In this way the student is shown how to discover what the form of a locus is. *Second,* he is given practice in exact description of the locus. *Third,* he proves his statements to be correct.

Indirect Proof. — This is introduced after the student has become familiar with the use of direct proof. Applications of this kind of proof to everyday decisions and inferences are shown.

John C. Stone
Virgil S. Mallory

ACKNOWLEDGMENTS

The authors wish to acknowledge their indebtedness to Miss Eleanor Booher of the Oak Park-River Forest High School, Oak Park, Illinois, who tried out much of the new material in her classes, and to Mr. Charles Kloss, State Teachers College, Montclair, for assistance in preparing the answers; also to the following who made helpful suggestions: Professor I. N. Warner, State Teachers College, Platteville, Wisconsin; Miss Celia J. Hatch, La Salle, Illinois; Miss Emma Hesse, University High School, Oakland, California; Miss Theodosia Keeler, High School, Sycamore, Illinois; Miss Grace Dean Maynard, Theodore Roosevelt High School, Des Moines, Iowa; Miss Leolian Carter, Central High School, St. Joseph, Missouri; Dr. S. Helen Taylor, University High School, University of Illinois, Urbana, Illinois; Miss Lena E. Reynolds, Fullerton High School and Junior College, Fullerton, California; Mrs. Dorothy B. Langstreth, John Marshall High School, Los Angles, California; and Mr. E. R. Rike, Woodward High School, Toledo, Ohio.

CONTENTS

		PAGE
UNIT ONE.	Introduction	1
UNIT TWO.	Formal Demonstration; Analysis; Constructions; Indirect Proof	67
UNIT THREE.	Parallels and the Angles Formed by Them; Parallelograms	119
UNIT FOUR.	Circles	173
UNIT FIVE.	Angles Measured by Arcs of Circles; Concurrent Lines; Constructions	221
UNIT SIX.	Similar Polygons	277
UNIT SEVEN.	Areas of Polygons	343
UNIT EIGHT.	Regular Polygons	397
APPENDIX.	Review of Algebra; Syllabus of Propositions	447
INDEX		471

ABBREVIATIONS AND SYMBOLS

Ax., axiom.
Alt., alternate; altitude.
Bis., bisector.
Comp., complementary.
Cons., construction.
Cor., corollary.
Corr., corresponding.
Def., definition.
Ex., exercise.
Fig., figure.
Ext., exterior.
Hyp., hypotenuse.
Int., interior.
Opp., opposite.
Post., postulate.
Prop., proposition.
Rect., rectangle.
Rt., right.
St., straight.
Supp., supplementary.
$=$, is equal to; equals.

$\neq$, is not equal to.
$\sim$, is similar to.
$\cong$, is congruent to; congruent.
$>$, is greater than.
$<$, is less than.
$\parallel$, is parallel to; parallel.
$\perp$, is perpendicular to; perpendicular.
$\therefore$, therefore.
$\ldots$, and so on.
$\angle$, angle.
$\measuredangle$, angles.
$\triangle$, triangle.
$\triangle$s, triangles.
▱, parallelogram.
▱s, parallelograms.
▭, rectangle.
▭s, rectangles.
$\odot$, circle.
Ⓢ, circles.
$\frown$, arc.

s.a.s. = *s.a.s.* Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.

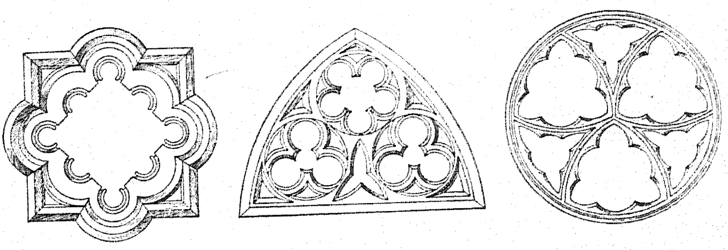
a.s.a. = *a.s.a.* Two triangles are congruent if two angles and the included side of one are equal, respectively, to two angles and the included side of the other.

s.s.s. = *s.s.s.* Two triangles are congruent if the sides of one are equal, respectively, to the sides of the other.

NEW PLANE GEOMETRY

UNIT ONE

INTRODUCTION



In beginning any new study you are interested in knowing what the study is about and why you should study it. These are fair questions and deserve an answer.

1. What is geometry? The dictionary will tell you that geometry is that branch of mathematics which studies the relations and properties of solids, surfaces, lines, and angles, and that plane geometry is the study of those figures which can be drawn on a **plane** or flat surface. History will tell you that the study of geometry began many centuries ago when man still lived in caves and used lines and circles for decorating the pottery he made. Later he built conical tents, igloos shaped like hemispheres, and laid out the floor of his house in a rectangle. Thus he became familiar with geometric figures.

2. The Egyptians. The people who lived in the fertile valley of the Nile in Egypt found a further use for geometry. Each year the river flooded their fields and it was necessary to stake them out again. They needed a knowledge of geometry to find areas and volumes, to lay out right angles at the corners of their fields, and to build their houses. Thus we see that early man used geometry in ornamentation and also in practical ways.

3. The Greeks. About twenty-five hundred years ago the Greeks began to study geometry in a more scientific way. They loved beauty and took delight in geometric ornamentation and in architectural effects. Thus they discovered many relations among geometric figures. They were not satisfied until they had *proved beyond a doubt* that the relations were true. In this they differed from the Egyptians. The Greeks studied geometry *as a method of thinking*. They decided that every educated person must learn to reason and to make precise accurate statements. Thus Plato (429 B.C.) put over the door of his school: *"Let no one unacquainted with geometry enter here."*

4. How geometry is used. You have seen that the Egyptians used geometry in their great surveying and building enterprises and that geometric figures are used in ornamentation and in architecture. Many of the pictures in this book show geometric figures used in quilt designs, linoleum patterns, wood carving, laces, church windows, etc., as well as many beautiful geometric forms found in nature. You will learn in this course how to make many of these designs. You will also learn how the ship captain or airplane pilot uses geometry in navigation, how the

draftsman uses it in drawing plans, and how the astronomer uses it in his study of eclipses and in determining the sizes and distances of celestial bodies. The engineer uses geometry in planning buildings, tunnels, bridges, and roads; the surveyor uses it in measuring and in laying out the plans of the engineer; the physicist and mathematician also use geometric principles in their daily work.

5. Why should you study geometry? Even though you may not plan to enter one of the vocations in which geometric figures and form are used directly, the study of geometry is important to you. It will teach you how to say what you wish in a precise, accurate way, how to express your arguments with exact logical reasoning, and how to prove the truth or falsity of statements. For these reasons alone every intelligent girl and boy should study geometry.

> **SOCRATES**
>
> A Greek philosopher who lived 469–399 B.C., demanded accurate definitions, clear thinking, and exact analysis. He developed a style of teaching which is often called the "Socratic Method." This method sought to reveal truth through a series of questions and responses. One of its important characteristics was its insistence on careful definition of terms.

6. Definitions. Why is it important to define carefully the terms we use? Arguments and misunderstandings sometimes result because of our failure to define carefully the things we talk about.

A class in geometry gave the following reasons why definitions are necessary.

Jane: So we will all be thinking of the same thing when the term is used.

Robert: To prevent misunderstanding.

Ruth: Unless we clearly understand what a term means we cannot use it intelligently.

Standish: If a term is properly defined, we know what kind of a thing it is and exactly what makes it different from other things which are somewhat like it.

Do you think that the answers are correct?

7. Properties of a good definition. "We must give in a definition the briefest possible statement of such qualities as are sufficient to distinguish the class from other classes."

W. S. Jevons: *Lessons in Logic*

While you probably know in a general way what such geometric figures as circles, triangles, etc., are, in the study of geometry it will be necessary for you to know the definition of such terms. There are a few geometric terms which represent concepts so elementary that they cannot be defined in simpler terms. This is true of our idea of a straight line.

A definition must be carefully thought out and should have the following properties:

1. *The terms used must be simpler than the term defined.*
2. *The term defined must be placed in its nearest class.*

Thus, you would not classify a drill as a piece of iron, but as a pointed cutting-tool.

3. *The difference between the term defined and other similar terms must be pointed out.*

For example, " A quadrilateral is a polygon *which has four sides.*"

4. *The definition must be reversible.*

Thus, " A triangle is a polygon " is not a definition because a polygon is not always a triangle.

EXERCISES

Tell which of the following are good definitions. Give reasons for your answers.

1. A *weasel* has thick fur.

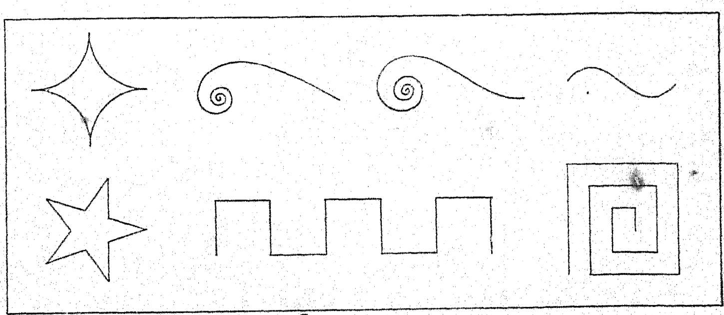
2. A *line* has length but no breadth or thickness.

3. A *horse* is a hoofed quadruped.

4. A *triangle* is a closed geometric figure formed by three straight lines intersecting in three distinct points.

5. A *circle* is a closed curve all points of which are equidistant from a point within called the center.

8. Geometric figures are made of straight and curved lines. The geometric figures on page 1 are designs for church windows and contain triangles, circles, and parts of circles.



STRAIGHT, CURVED, AND BROKEN LINES

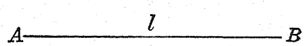
THALES

Lived in Miletus about 600 B.C. He is called the "Father of Geometry." His most noted scholar was Pythagoras.

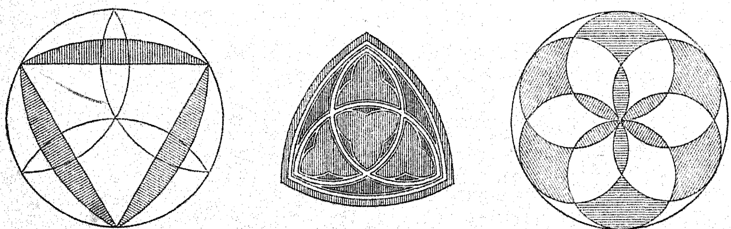
9. Kinds of lines. A true **straight line** has length, but no width or thickness. You know that you can represent a straight line by a stretched string and that you can draw a representation of a straight line by using a ruler or straightedge as a guide.

A straight line has no definite length. A definite part of a straight line is called a **line segment.** When there is no chance for misunderstanding we speak of a straight line as a line and of a line segment as a segment.

Points are represented by capital letters; lines by small letters. Thus we may speak of the line AB or of the line l.



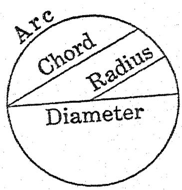
A **broken line** is made up of connected parts of straight lines. A **curved line** is a line no part of which is straight.



DESIGNS MADE FROM CIRCLES

10. Circles. A **circle** (symbol ⊙) is a closed curve all points of which are equally distant from a point within called the **center.**

A line segment from the center to the circle is called a **radius.**



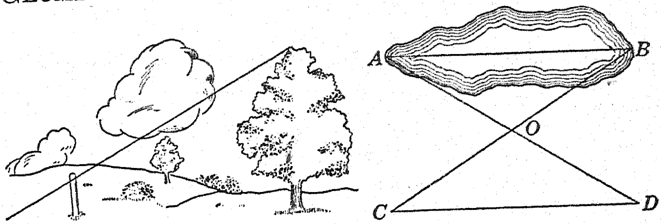
The **diameter** is a line segment through the center terminated by the circle.

A diameter is equal to two radii.

A **chord** is a line segment whose end points are on a circle.

Any part of a circle is called an **arc** (symbol $\frown$).

Geometry in Finding Heights and Distances



Geometry is used in many practical problems in finding heights and distances.

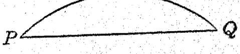
EXERCISES

1. Mark a point P on your paper. How many straight lines can you draw through P?

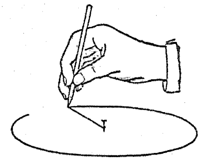
2. Mark two points P and Q. How many straight lines can be drawn through both P and Q?

3. Draw a straight line segment AB about 2 in. long. Mark its mid-point. How many mid-points can a line segment have?

4. Two points P and Q are connected by a straight line and a curved line. Which line is the shorter?



5. Two straight lines AB and CD intersect in a point P. Can the lines have another point of intersection, Q?

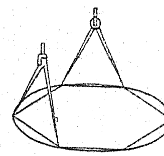


6. Draw a circle. Call its center O. Draw three radii (rā′di-ī). Are they all equal? Are all diameters of a circle equal?

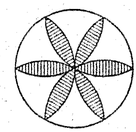
7. Write statements of your conclusions in Ex. 1–6. See if any of your conclusions are like those on page 13.

---

8. Draw a circle with any point as center and any convenient radius. Divide the circle into six equal arcs by stepping off the radius on the circumference. Join the successive points of division. The figure formed is called **a hexagon.** By connecting the alternate points of division an **equilateral** triangle is formed. What points can be connected to form an isosceles triangle?



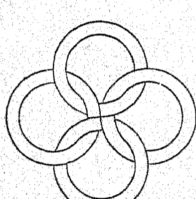
9. Mark off six equidistant points on a circle. With each point as center, and with the same radius, draw arcs as in the figure. This figure is the basis of many circular designs.

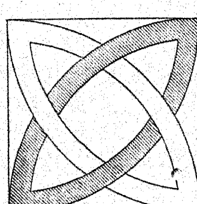


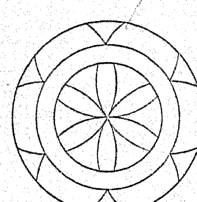
10. The figure at the right (bottom of page) is based on the division of a circle into six equal arcs. Make a copy of it, making the diameter of the smallest circle 2 in.

11. Copy the middle figure, first drawing a square whose side is 2 in. Use a drawing triangle or any square corner to make the corners of your square.

*12. Copy the figure at the left in a 2-inch square. The radius of each of the larger circles is $\frac{1}{2}$ in. How, by marking off $\frac{1}{2}$-inch segments on the sides of the square, can you locate their centers?

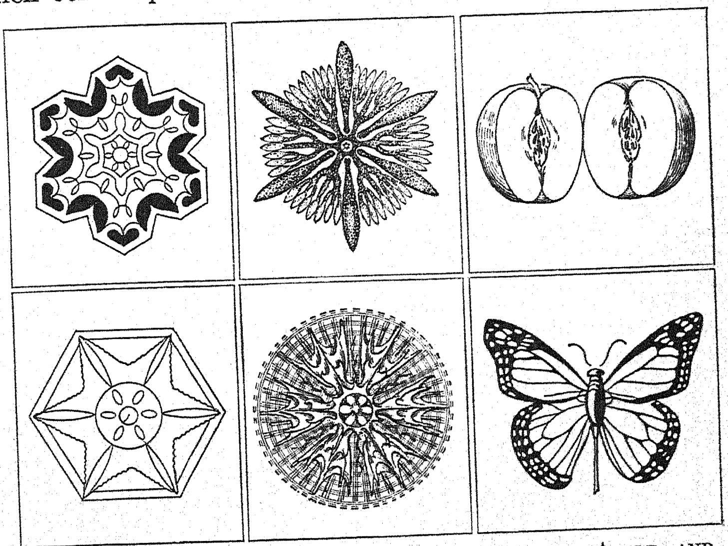






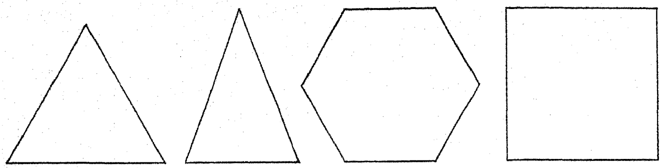
11. Symmetry. Many of the geometric forms in nature are beautiful because their parts are balanced. When the parts of a figure are balanced with respect to a point, a line, or a plane, the figure is said to possess **symmetry.**

Notice the symmetry in the forms below and on page 10. A line through the center of the butterfly would divide it into two parts of the same shape. The line is called the **axis of symmetry.** The snowflakes are symmetrical about their central point. Have they also axes of symmetry?



SNOW CRYSTALS, BITS OF CORAL, THE HALVES OF AN APPLE, AND THE WINGS OF A BUTTERFLY SHOW SYMMETRY

12. Polygons. Any closed figure bounded by line segments is called a **polygon.** The line segments are called the *sides* of the polygon and the points in which they intersect the *vertices* (singular, *vertex*) of the polygon.



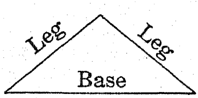
POLYGONS

13. A **triangle** is a polygon having three sides. Thus a triangle (symbol $\triangle$) has three sides and three vertices.

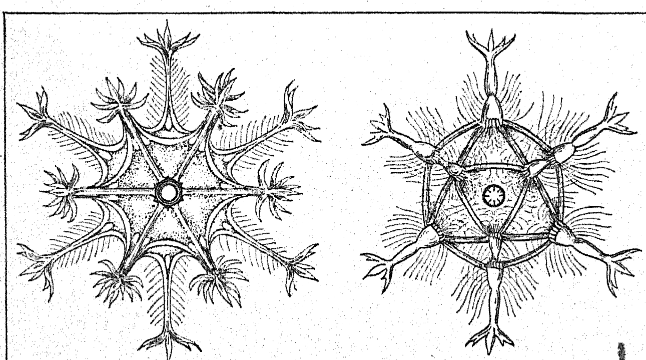
14. Kinds of triangles. Triangles can be classified according to the relative lengths of the sides.

An **equilateral** triangle is a triangle that has all of its sides equal.

An **isosceles** triangle is a triangle with two sides equal. The equal sides of an isosceles triangle are called the **legs.** The other side is called the **base.**



A **scalene** triangle has no two sides equal.



MICROSCOPIC ANIMALS

NOTICE THAT THESE FIGURES HAVE AXES OF SYMMETRY. THEY ARE ALSO SYMMETRICAL ABOUT THE CENTER POINTS.

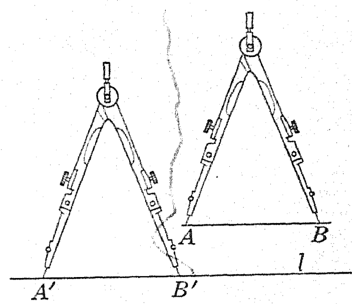
Geometric Constructions

15. Geometric constructions. We shall use the word construct in a technical sense, meaning to draw with the aid only of the straightedge and compasses. The straightedge will be used to draw straight lines. The compasses will be used to draw circles and to transfer lengths.

16. To copy a line segment. To copy the line segment AB proceed as follows:

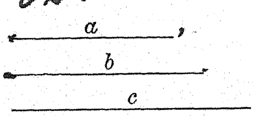
1. Draw a straight line of indefinite length on your paper. Call it l. Use your straightedge to draw the line but not to measure the length of it. Select any point on l and label it A'.

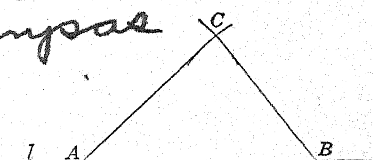
2. To get the length of AB, place one point of your compasses on A and adjust the compasses until the other point falls on B. Then with one point of your compasses at A' mark the position of the other point on l and call it B'. Thus you have copied the segment AB on l. Call this segment $A'B'$ (read *A prime, B prime*).



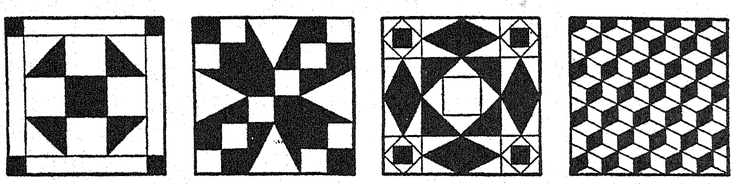
17. To construct a triangle when three sides are given.

Use the compass





1. On a line l of indefinite length take $AB = c$. 2. With A as center and b as radius draw an arc. 3. With B as center and a as radius draw an arc intersecting the first arc at C. 4. Draw AC and BC. 5. ABC is the required triangle.



QUILT DESIGNS

EXERCISES

1. Construct a line segment equal in length to a and b:
$a + b$; $3a + b$; $b - a$; $4a - 2b$.

2. Construct a line segment of length $\frac{3}{4}$ in.; $1\frac{1}{4}$ in.; $\frac{3}{8}$ in.; $1\frac{7}{8}$ in.

3. Construct a triangle with sides 2 in., $2\frac{1}{4}$ in., and $1\frac{3}{4}$ in.

4. Construct an equilateral triangle with each side equal to 2 in.

5. Construct an isosceles triangle with base 1 in. and equal sides each $1\frac{3}{4}$ in.

6. Do you think that two triangles are congruent (con′gruent, that is, exactly the same) if they have the sides of one equal respectively to the sides of the other? Test by constructing two triangles and cutting them out.

7. Can you construct a triangle with sides 1 in., 2 in., and 3 in.?

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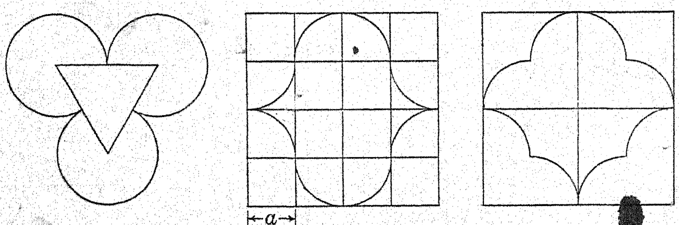
8. Make a statement about the relation which must exist between the sides of a triangle to make its construction possible.



9. Copy the designs shown above. The first is a window design. The other two are tile designs. In drawing the second one lay off a segment a four times. Then complete the square as in Ex. 11, § 10.

18. Fundamental principles. We shall assume that the following statements are true. Such assumptions are called **postulates.** We shall use them frequently to justify statements that we shall make.

Read these postulates over carefully. You need not memorize them now. You will find all of the postulates we shall use in this course listed in the *Appendix*.

POSTULATE 1. *A straight line can be produced (drawn) to any required length.*

POSTULATE 2. *Two straight lines cannot intersect in more than one point.*

POSTULATE 3. *Through two given points one and only one straight line can be drawn.*

POSTULATE 4. *The length of the line segment connecting two points is the shortest distance between them.*

POSTULATE 5. *A circle may be drawn with any point as center and with any line segment as radius.*

POSTULATE 6. *All radii and all diameters of the same circle or of equal circles are equal.*

POSTULATE 7. *A geometric figure may be moved without changing its size or shape.*

19. Applications of the postulates. The following applications follow directly from the postulates just given.

EXERCISES

1. Mark two points A and B on your paper and connect the points with a straight line. Which postulate is illustrated by the fact that AB can be produced in either direction as far as you wish?

2. Use the following method to test the accuracy of your ruler: Mark along the edge, then reverse the ruler, placing it so that it touches the line at two points, and draw a second line. What will show whether the ruler is true or not?

3. If the edge of your ruler is straight and is placed so that it touches a straight line at two points, will the ruler touch the line at every point of the ruler? Upon what property of the straight line in § 18 is the answer based?

4. Use the following method to test the straightness of a segment AB. Place over it a piece of thin paper, through which the segment may be seen, and trace it, marking points A' and B' over A and B, respectively. Then reverse the paper so that point A' falls upon point B and point B' upon point A. Is the segment straight? Explain.

5. Explain which postulates are illustrated by:

a. Two sights on a gun.

b. A cord stretched taut is straight.

c. Fence stakes can be set in a straight line by sighting over two stakes to locate the position of a third.

d. A surveying monument is marked with a cross.

6. How can you locate a point known to be on each of two lines?

7. What is meant by "Two points determine a straight line"?

8. What is meant by "Two intersecting straight lines determine a point"?

9. If two straight lines are in the same plane do they necessarily determine a point?

10. Mark a point O. With O as center and a radius of $1\frac{1}{2}$ in. draw a circle. Which postulate is illustrated?

11. On circle O (Ex. 10) mark any two points A and B. Draw OA and OB. Why does $OA = OB$? Produce AO and BO through O to meet the circle at C and D, respectively. Why does $AC = BD$?

---

12. How many straight lines can be drawn through three points, if the points are not all in the same line? How many straight lines can be drawn through four points? Five points?

13. Show that the formula for the number of straight lines that can be drawn through n points, no three of which are in a straight line, is $L = \frac{n}{2}(n - 1)$. Test for $n = 3, 4, 5$, and 6.

14. In general, in how many points will three lines intersect? Four lines? Five lines? Is the formula $P = \frac{n}{2}(n-1)$?

15. Can a straight line (indefinitely long) intersect a circle in only one point? Can a straight line intersect a circle in more than two points?

16. If treasure is buried so that it is on each of two distinct circles, show by a drawing its possible locations.

EUCLID

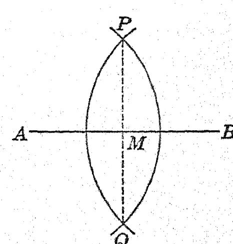
Was librarian in the University at Alexandria, Egypt, about 300 B.C. He collected all that was known about geometry and arranged it in logical form.

MID-POINT OF A SEGMENT

20. A line segment is **bisected** when a point divides it into two equal parts. The point is called the **mid-point** of the segment.

21. To bisect a segment. The directions for bisecting a segment AB are:

1. With A as center and with any radius more than half AB, draw an arc.
2. With B as center and with the same radius, draw an arc intersecting this arc in P and Q.
3. Draw PQ intersecting AB in M.
4. M is the required mid-point.



Test the accuracy of your construction by using your compasses to compare the segments AM and MB. Later in this course we shall *prove* that $AM = MB$.

22. POSTULATE 8. We shall assume the following: *A line segment has one and only one point of bisection.*

ARCHIMEDES

Lived in Syracuse, 225 B.C. While he used many practical applications of geometry, he liked best to discover new geometric truths.

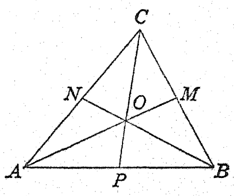
EXERCISES

1. Draw line segments in several different positions. Construct the bisector of each. Test the accuracy of each construction with your compasses.

2. Divide a segment into four equal parts. Can the method of bisecting a segment be used to divide a segment into five equal parts? Into six? Explain.

3. Can you write a formula which will express the number of equal parts into which a segment can be divided by repeated bisections?

4. Construct a triangle ABC with sides 3 in., $2\frac{1}{2}$ in., and $2\frac{1}{4}$ in. Find the mid-points M, N, and P of the sides and draw AM, BN, and CP. These three lines are called the **medians** of the triangle. (A median of a triangle is a line segment from any vertex to the mid-point of the opposite side.) If your construction is made carefully the medians will pass through a common point O.



Test with your compasses and answer the following questions:

Is $AM = BN = CP$? Is $AO = 2\,OM$? $BO = 2\,ON$? $CO = 2\,OP$?

5. Construct an isosceles triangle and its three medians. Do any of the medians seem to be equal?

6. Do the medians of an equilateral triangle seem to be equal?

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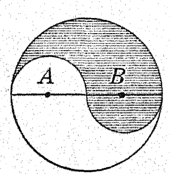
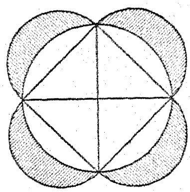
7. Draw a circle and divide its diameter into four equal parts. With A and B as centers construct semicircles as shown. Is the area of the circle divided into two equal parts? Why?



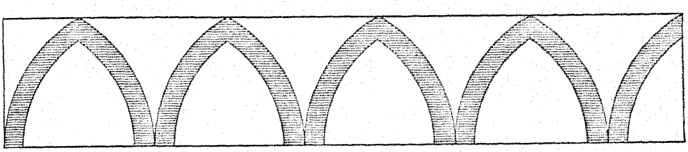
8. An old problem: A man traveled from A to B by first going half the distance from A to B; then going half the remaining distance. If he continued this, each time

going half the remaining distance toward B, would he ever reach his destination?

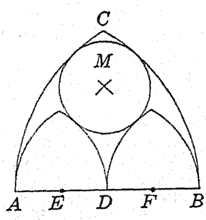
9. Draw a circle and two perpendicular diameters (see Ex. 11, § 10). Connect the extremities of the diameters to form a square. On each side of the square as diameter draw a semicircle. Shade the four outer crescents.

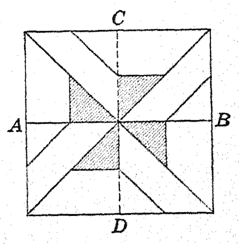


NOTE. — This construction was first made by Hippocrates, a Greek mathematician who lived about 460 B.C. He proved that the sum of the areas of the shaded crescents was equal to the area of the square. You will learn how to prove such facts later in this course.



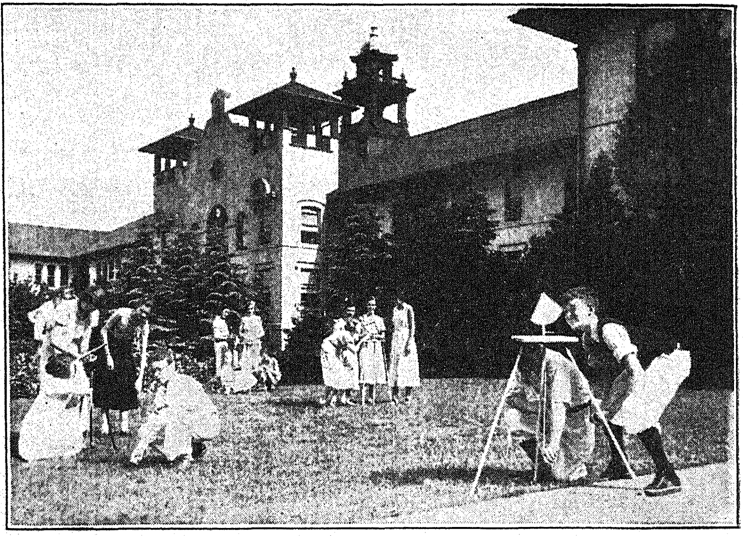
10. Draw a design like the one above. First divide a line segment into eight equal parts. With the ends of these parts as centers, and with a radius equal to two of the parts, draw arcs as shown in the figure. In drawing the smaller arcs use the same centers.





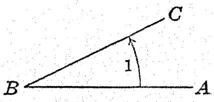
11. The drawing at the left shows the design of a Gothic window. Arc BC is drawn with radius AB and center A. Arc AC is drawn with radius AB and center B. The small arcs are drawn with radii equal to AD and centers at A, D, and B. The center M of the circle is found by drawing arcs with centers A and B and radii equal to AF.

12. Copy the drawing at the right. It represents parquet flooring. Each side of the square is divided into four equal segments, and the segments within the square are bisected.

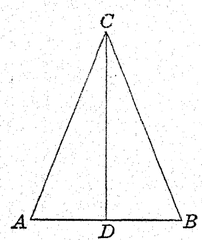


GEOMETRY IS USED IN SURVEYING

23. An **angle** ($\angle$) is the figure formed by two straight lines drawn from the same point. B is the **vertex** and AB and CB are the **sides.**



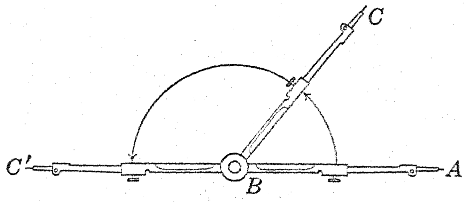
The size of the angle depends on the amount the side AB must revolve about B to take the position CB. The size does not depend on the length of the sides.



The angle may be read $\angle B$, $\angle 1$, or $\angle ABC$. Notice that when three letters are used the middle one denotes the vertex of the angle.

A single letter can be used to denote an angle only when there is no ambiguity. Thus, in the

second figure, $\angle C$ does not tell which of the three angles ACD, DCB, or ACB is meant.



If you hold one arm of your compasses (AB) fixed in one position and revolve the arm BC in a counterclockwise direction (opposite to the direction in which the hands of a clock move) the angle ABC becomes larger.

We sometimes call the fixed line AB the **initial side** and the line CB that revolves, the **terminal side** of the angle.

If the terminal side revolves until it takes the position BC' so that ABC' forms a straight line, then the angle ABC' is a straight angle.

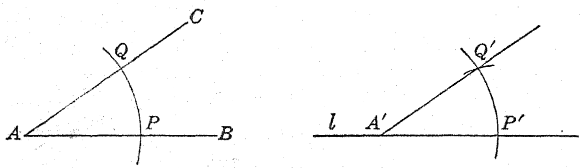
The sizes of two angles may be compared by placing one angle on the other so that their vertices coincide and a side of one falls along a side of the other. If, then, their other sides lie in the same direction, the angles are equal.

Two angles are equal if they can be made to coincide.

EGYPTIAN MATHEMATICIANS

Were called Harpedonaptae, or Rope Stretchers. This is because they used a rope marked off into lengths of 3, 4, and 5 units in making the corners of their buildings square.

24. To copy an angle. To copy $\angle BAC$ at point A' on line l, proceed as follows:

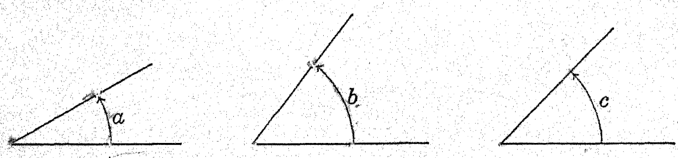


1. With A as center and any convenient radius draw an arc intersecting BA and CA in P and Q, respectively.
2. With A' as center and the same radius draw an arc intersecting l at P'.
3. With P' as center and with a radius equal to PQ mark the point Q' on the arc drawn in (2).
4. Draw $A'Q'$. Angle $P'A'Q'$ is the required angle.

Test the accuracy of your construction by cutting out $\angle P'A'Q'$ and placing it on $\angle PAQ$. Are there any possible inaccuracies in such a test? We shall later prove that $\angle P'A'Q' = \angle PAQ$.

EXERCISES

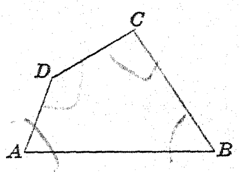
1. Construct an angle equal to a given angle and test your construction by using tracing paper.



2. Construct an angle equal to $\angle a + \angle b$; $\angle a + \angle b + \angle c$; $2\angle a + \angle b$; $\angle b - \angle a$.

3. Draw a triangle ABC and construct an angle equal to $\angle A + \angle B + \angle C$. How large does the sum seem to be?

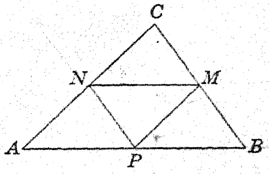
NOTE. — Make the triangle large enough so that you can copy the angles easily.



4. Draw a quadrilateral $ABCD$ and construct an angle equal to the sum $\angle A + \angle B + \angle C + \angle D$. How large does this sum seem to be?

Repeat with a different shaped quadrilateral. Does the sum seem to be the same?

5. Repeat Ex. 4, using a figure having five sides (pentagon). Does the sum of the angles seem to depend on the number of sides?



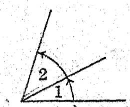
6. Draw a triangle ABC and bisect the sides at M, N, and P. Draw MN, MP, and NP, thus forming four other triangles. Using your compasses compare the size of the following angles:

(a) $\angle A$, $\angle MNC$, $\angle BPM$, and $\angle NMP$.
(b) $\angle B$, $\angle CMN$, $\angle NPA$, and $\angle PNM$.
(c) $\angle C$, $\angle PMB$, $\angle ANP$, and $\angle MPN$.

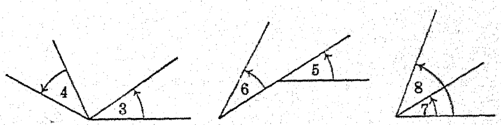
7. Congruent (con′gruent) triangles are triangles that are the same in *shape* and in *size*. **Similar** triangles have the same *shape* but not necessarily the same *size*. Do any of the triangles in Ex. 6 seem to be congruent? Do the small triangles seem to be similar to triangle ABC?

AHMES

Who lived 1650 B.C. left us the earliest record we have of Egyptian mathematics. The papyrus he wrote was copied from one written 2350 B.C.



ADJACENT ANGLES

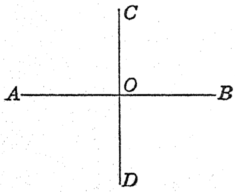


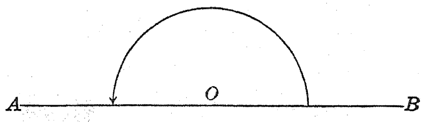
THESE ANGLES ARE NOT ADJACENT

25. Adjacent angles are angles that have the same vertex and a common side *between* them.

Why are $\angle$s 3 and 4 not adjacent? Why are $\angle$s 5 and 6 not adjacent? $\angle$s 7 and 8?

26. Perpendicular. If two lines intersect so that the adjacent angles formed are equal, the lines are said to be **perpendicular** ($\perp$).

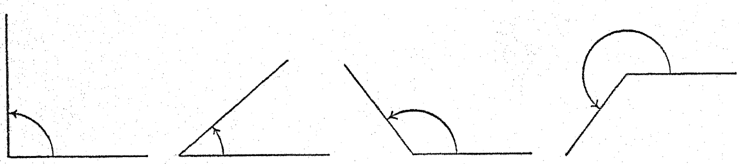




STRAIGHT ANGLE

Thus $CD \perp AB$.

27. Straight angle. A straight angle is an angle whose sides extend in opposite directions from the vertex and form a straight line.



RIGHT ACUTE OBTUSE REFLEX

28. Right angle. Each of the angles formed by perpendicular lines is called a **right angle.**

Hence a straight angle contains two right angles, or a right angle is half a straight angle.

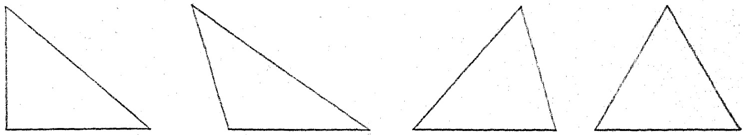
29. Other angles. An **acute** angle is an angle less than a right angle.

An **obtuse** angle is an angle greater than a right angle and less than a straight angle.

A **reflex** angle is an angle greater than a straight angle and less than two straight angles.

Reflex angles and angles greater than two straight angles

are used in higher mathematics but you will not need them in this course.



RIGHT OBTUSE ACUTE EQUIANGULAR

30. Triangles. You have classified triangles according to their sides as scalene, isosceles, and equilateral. Triangles are also classified according to their angles.

A **right triangle** is a triangle that has one right angle.

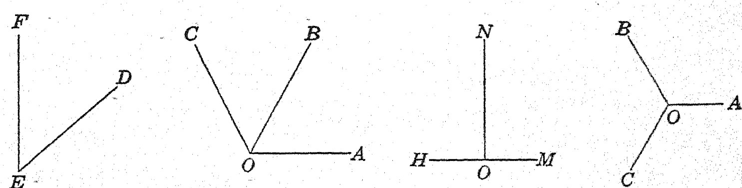
An **obtuse triangle** is a triangle that has one obtuse angle.

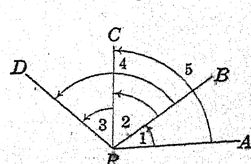
An **acute triangle** is a triangle all of whose angles are acute angles.

An **equiangular triangle** is a triangle all of whose angles are equal.

EXERCISES

1. Using three letters read each of the angles below. Tell what kind of angle each seems to be.





2. What are the sides of $\angle 1$? What are the sides of $\angle 2$? Of $\angle 3$? Of $\angle 4$? Of $\angle 5$?

3. Name each of the angles, using the letters. Notice that the vertex is the middle letter.

4. What is the common vertex?

DPA BPA
CPA CPB
DPB DPC

5. How many pairs of angles can you find that have a common side? There are eight pairs in all. Name them.

6. Four of these pairs of angles in Ex. 5 are adjacent. Which are they?

7. Name the four pairs having a common side that are not adjacent.

8. Draw in your notebook two angles that have the same vertex and no common side.

9. Draw two angles that have a common side and different vertices.

10. Draw two angles with the same vertex and a common side, but the common side not between the angles.

11. Draw two adjacent angles.

12. Draw two angles. Name one $\angle ABC$ and the other $\angle DEF$. Cut out or trace $\angle DEF$ and place it upon $\angle ABC$ so that E falls upon B and ED falls along BA. If EF falls within $\angle ABC$, which angle is greater? What angle shows the difference? If EF falls along BC, what is known of the two angles?

13. Does the size of an angle depend on the length of its sides?

14. What kind of angle is the smaller angle formed by the hands of a clock at 1 o'clock? At 3 o'clock? At 5 o'clock? What kind of angle is formed at 6 o'clock?

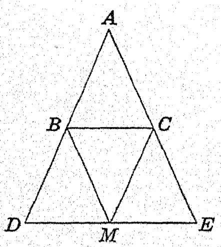


FIG. 1

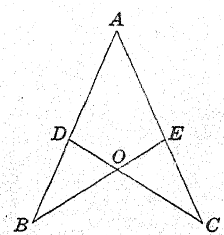


FIG. 2

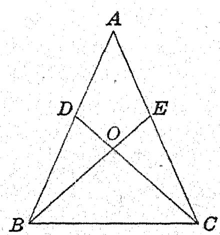


FIG. 3

15. In Fig. 1 name five triangles.

16. In Fig. 2 name four triangles.

17. In Fig. 3 name eight triangles.

18. Using the method of § 17 construct (Fig. 2) two triangles with sides equal, respectively, to the sides of triangles ABE and ACD.

Cut the triangles out of paper and place them together as in the figure. Notice the overlapping.

19. As in Ex. 18 construct (Fig. 3) three triangles with sides equal, respectively, to the sides of triangles ABE, ABC, and ACD. Place them together as in the figure and notice the overlapping.

31. Postulates. We shall accept as true the following general principles about angles.

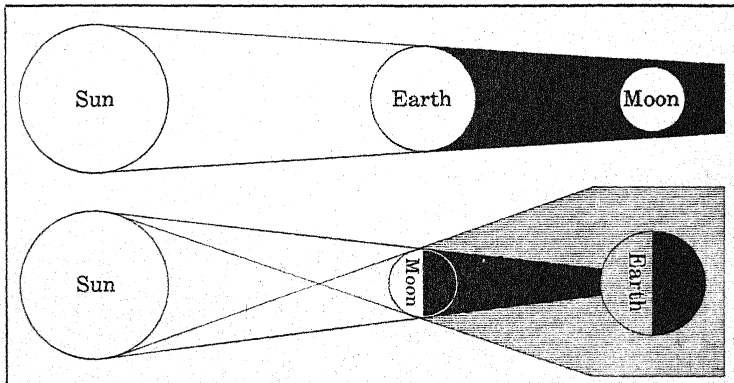
POSTULATE 9. *All right angles are equal.*

Since a straight angle contains two right angles (§ 28),

All straight angles are equal.

32. An angle is bisected when a line divides it into two equal parts. (The line must pass through the vertex.)

POSTULATE 10. *An angle has one and only one bisector.*



GEOMETRY IS USED IN ASTRONOMY

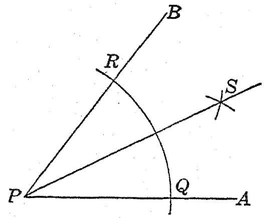
Geometry is used in astronomy to determine the positions of the stars, in the study of eclipses, and to find the latitude of points on the earth.

33. To bisect an angle.

1. With P as center and any radius draw an arc intersecting AP and BP in Q and R, respectively.

2. With Q and R as centers and a radius more than half QR draw arcs intersecting in S.

3. Draw PS. PS is the required bisector.



Test the accuracy of your construction by cutting out and folding along the bisector.

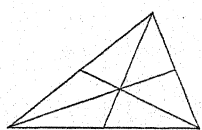
EXERCISES

1. Bisect an acute angle, a right angle, and an obtuse angle. In each case test the accuracy of your construction by cutting out and folding along the bisector.

2. Divide an angle into four equal parts. Can the method of bisecting an angle be used to divide an angle into five equal parts? Into six? Into eight?

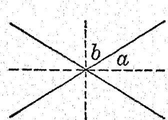
3. Make a formula about the number of equal parts into which an angle can be divided by repeated bisection.

4. Draw a triangle and bisect each of its angles, producing the bisectors until they meet the opposite side. Do the bisectors seem to be equal? Make a separate statement about equilateral, isosceles, and scalene triangles.



5. In Ex. 4 do the bisectors of the angles seem to bisect the opposite sides? Make a separate statement about equilateral, isosceles, and scalene triangles.

6. Draw two intersecting lines and bisect each of the four angles formed. Do the bisectors seem to be perpendicular?



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7. Bisect a straight angle. From the result make a statement about constructing a perpendicular to a line from a point on the line.

8. Can you tell why the statements you made in connection with Ex. 4 and Ex. 5 are not *proofs?*

9. What is the difference between a *proof* and a conclusion drawn from *experiment?*

***10.** Can you *prove* that the bisectors formed in Ex. 6 are perpendicular, that is, $\angle a + \angle b = 1$ right angle?

***11.** The following method is used by boy scouts for determining a *meridian* or true north and south line. A stake is driven into the ground and its shadow is marked on the ground about two hours before noon and again the same number of hours after noon. The bisector of the angle is the meridian. Explain why this is so.

NOTE. — This can be done without a watch provided the stick is vertical. At noon the shadow is shortest. It has the same length at equal times before and after noon.

MEASURING ANGLES

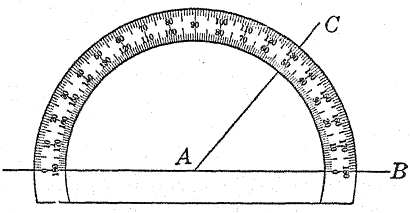
34. Unit of measure. The unit that is most frequently used in measuring angles is called a **degree.** It is one 360th part of the entire angular magnitude about a point. Thus, in terms of this unit, a right angle contains 90 degrees, written 90°. Since a straight angle is equal to two right angles, it contains 180°.

THE DEGREE

It is thought that the division of a circle into 360 equal parts and hence the use of a degree as the unit in measuring angles can be traced to the Babylonian use of the number *sixty* in writing numbers. Thus influenced, the Greek astronomer Ptolemy (150 A.D.) divided the radius of the circle into 60, and the diameter into 120, equal parts. Then, in the belief that the circumference was 3 times the diameter, he divided the circumference into 3×120 or 360 equal parts.

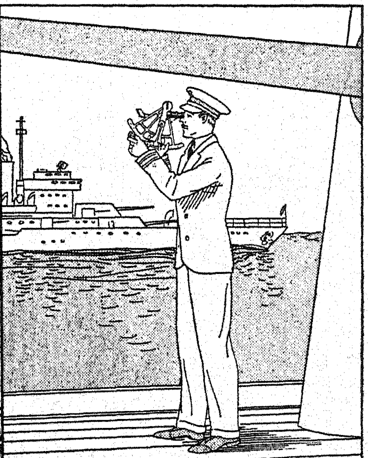
A degree is divided into 60 **minutes** (written 60′), and a minute into 60 **seconds** (written 60″).

An instrument used to measure the number of degrees in an angle is called a **protractor.** To use it for measuring $\angle BAC$, the center of the protractor is placed over A, the vertex of the angle, and the protractor is turned until the zero point of the scale falls on the side AB or AB extended. The number of degrees in the angle is read at the point on the scale where AC or AC extended crosses the scale. Thus, in the drawing, $\angle BAC = 50°$.



INSTRUMENTS FOR MEASURING ANGLES

USING A TRANSIT

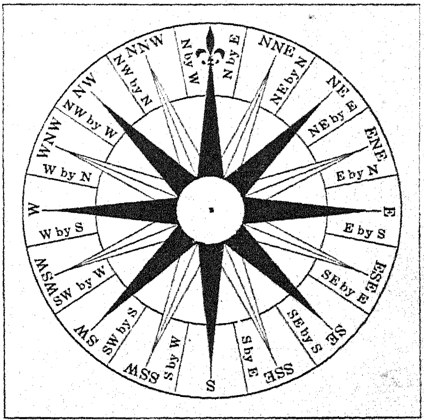


USING A SEXTANT

Surveyors, astronomers, and navigators use instruments which measure angles with great precision. The surveyor's transit has a telescope (for sighting distant points) mounted so that both vertical and horizontal angles can be read on protractors carefully placed for that purpose. A simple homemade transit * is shown on page 30.

The navigator uses an instrument called a sextant for measuring the angle of elevation of the sun or of a star. From this information he is able to calculate his position in latitude and longitude. The sextant consists of a telescope mounted on a protractor.

The mariner's compass shows the division of the entire angular magnitude about a point into 32 equal parts called the points of the compass. Hence each point differs from the preceding one by $11\frac{1}{4}°$. Each of these points has a name as indicated in the picture. To "box the compass" means to name these points in order. Generally directions at sea are indicated, however, by giving the number of degrees from north in a clockwise direction. Thus a course of 90° would be an easterly course; one of 225° would be southwesterly; etc. On land directions are generally referred to the four cardinal points. Thus N 20° E means a direction 20° east of north; S 45° W means a southwesterly direction; etc.

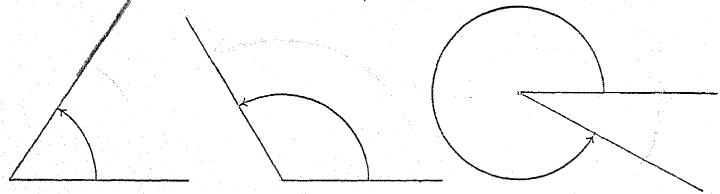


MARINER'S COMPASS

* For method of constructing the homemade transit see Stone-Mallory *Mathematics for Everyday Use* (Benj. H. Sanborn & Co.), p. 519.

EXERCISES

1. Copy on your paper angles like those below and measure each with your protractor. Explain the method in each case.



2. Draw any angle and bisect it as in § 33. Test the accuracy of your construction with your protractor.

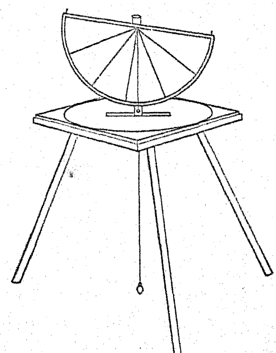
3. Draw an angle and bisect it by using your protractor.

4. Draw an obtuse angle. Copy the angle by using your protractor.

5. Using the protractor draw a right angle.

6. Draw angles of 15°, 30°, 45°, 75°, 120° and 165°.

7. How many degrees between the hands of a clock at 1 o'clock? At 3 o'clock? At 6 o'clock? At 9 o'clock? At 12 o'clock?



HOMEMADE TRANSIT

NOTE. — In each case give both angles.

~~**8.**~~ Draw a large triangle and measure each of the angles. What does the sum seem to be?

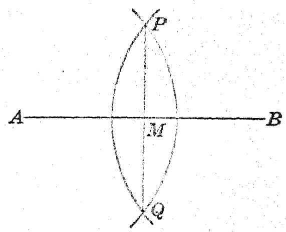
9. One angle contains 64° 24′ 35″ and another contains 47° 28′ 42″. Find their sum.

10. Subtract 23° 48′ 50″ from 90°; from 62° 25′.

CONSTRUCTIONS

35. The perpendicular bisector of a segment is the line perpendicular to the segment at its mid-point.

36. To construct the perpendicular bisector of a line segment. The construction is the same as that given in § 21 for bisecting a line segment.



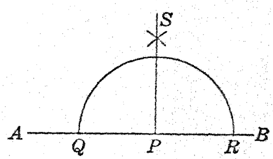
Is $PM \perp AB$? Measure with your protractor.

Is $AM = MB$? Test with your compasses.

Could you tell by using your protractor if angle BMP were a very little, say one minute, less than a right angle?

Later in the course we shall be able to prove that $PM \perp AB$ and that $AM = MB$.

37. To construct a perpendicular to a line at a point on the line.



If APB is a straight line, what kind of angle is $\angle BPA$ (§ 27)?

If you bisect $\angle BPA$, what two angles will be formed?

Will SP be $\perp$ to AB at P? Why (§ 26)?

Write out the construction, using the form given in § 21.

EXERCISES

1. Draw a straight line and construct its perpendicular bisector.

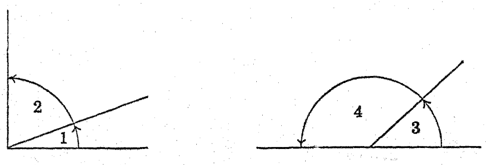
2. Draw an acute triangle and construct the perpendicular bisector of each side. Do the perpendicular bisectors seem to be *concurrent;* that is, have a common point of intersection?

3. Construct the perpendicular bisectors of the sides of an obtuse triangle. Does the point of intersection of the perpendicular bisectors lie inside or outside of the triangle?

4. Construct the perpendicular bisectors of the sides of a right triangle. At what point do the perpendicular bisectors seem to intersect?

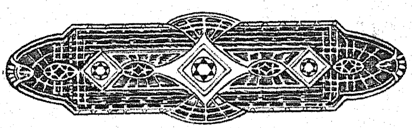
Complements and Supplements

38. Two angles whose sum is a right angle are said to be **complementary,** or one angle is the **complement** of the other, as $\angle 1$ and $\angle 2$.



Two angles whose sum is a straight angle are said to be **supplementary,** or one angle is the **supplement** of the other, as $\angle 3$ and $\angle 4$.

Angles need not be adjacent to be complementary or supplementary.



Geometric Designs in Modern Jewelry

EXERCISES

1. What is the complement of an angle of 60°? Of 45°?

2. What is the complement of an angle of 48° 24′ 15″? Of 62° 48″? Of x°?

3. Construct the complement of a given angle ABC by use of straightedge and compasses; by use of straightedge and protractor.

4. What is the supplement of an angle of 120°? Of 80°?

5. What is the supplement of an angle of 93° 40′ 30″? Of 133° 45″? Of x°?

6. Construct the supplement of a given angle ABC.

7. What is the angle which equals five times its complement?

Hint. — Let $5x$ equal the number of degrees in the angle. For Ex. 7, 8, and 9 see the *Review of Algebra* in the Appendix.

8. What is the angle which equals eight times its supplement?

9. Two angles are complementary. Three times the smaller is 5° less than twice the larger. What are the angles?

10. If two angles are complementary, can one of them be obtuse? Explain.

11. If two angles are supplementary, can both be obtuse? Can one be? Can both be acute? Can one be? Explain.

12. How large is an angle which is equal to its own complement?

13. How large is an angle which is equal to its own supplement?

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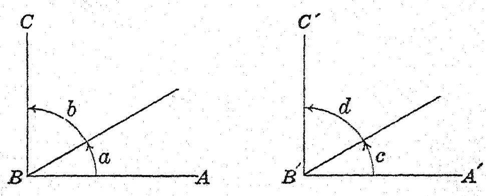
14. Find the difference between the supplement and the complement of an angle of 30°; of 45°; of 60°.

15. Repeat Ex. 14 for an angle of 10°; of 25°; of 80°.

*16. Find the difference between the supplement and the complement of an angle of $x°$.

*17. Complete the statement: The difference between the supplement and the complement of a given angle is an angle of ——— degrees. Why does your work in Ex. 16 *prove* the truth of this statement, while that in Ex. 14 and 15 does not?



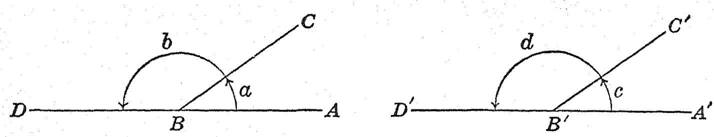
39. Complements of equal angles. In the figures above, if angles ABC and $A'B'C'$ are right angles, angles a and b are complementary, and angles c and d are also complementary. Why?

How many degrees in $\angle b$ and in $\angle d$ if $\angle a = \angle c = 30°$? If $\angle a = \angle c = 45°$? If $\angle a = \angle c = 60°$?

If $\angle a = \angle c = x°$, how many degrees in $\angle b$? In $\angle d$?

Thus we have *proved*

Equal angles have equal complements.



40. Supplements of equal angles. In the figures $\angle ABC = \angle A'B'C'$. Also $\angle ABD$ and $\angle A'B'D'$ are straight angles. Hence $\angle b$ is the supplement of $\angle a$, and $\angle d$ is the supplement of $\angle c$.

How many degrees in $\angle b$ and in $\angle d$ if $\angle a = \angle c = 50°$? If $\angle a = \angle c = 70°$? If $\angle a = \angle c = 80°$?

If $\angle a = \angle c = x°$, how many degrees in $\angle b$? In $\angle d$?

Thus we have *proved*

Equal angles have equal supplements.

41. The statements in §§ 39 and 40 are easy *theorems* which we have proved informally. A **theorem** is a statement of a truth to be proved.

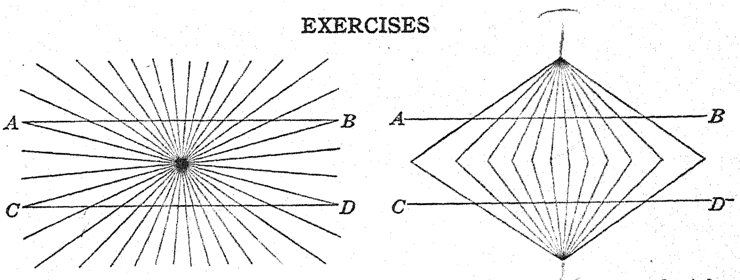
The Need for Logical Proof

42. Drawing conclusions from experiment. In some of the problems you have studied so far in this course you have drawn conclusions from experiment. Thus in constructing a bisector of a line segment (§ 21) you tested with your compasses to see if the parts formed seemed to be equal. You also tested the accuracy of the construction for copying an angle and for bisecting an angle. You have seen that this method of experimentally testing results is not exact. That is, while you are able to say that the method used *seems* to give the correct construction, you have not *proved* that the construction will be correct in every case.

No method that depends on measurement can be absolutely exact, for all measurements are subject to error.

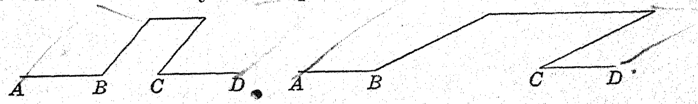
43. Optical illusions. Conclusions drawn from observation are also subject to error. Objects seen from a distance such as automobiles and people seen from a window in a tall building appear much smaller than they really are. The stars appear to be mere points of light but astronomers tell us they are suns enormous in size. Some examples of optical illusions are given in the exercises that follow.

EXERCISES

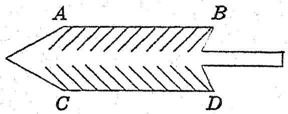


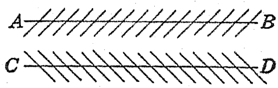
1. Are AB and CD straight or curved lines? Make your decision before you test with a straightedge in both figures.

2. Look at the figures and guess which is longer, a or b? EF or GH? Then test with your compasses.

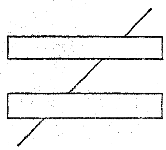


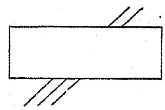
3. Is CD a continuation of AB? Test in both figures as before.





4. Are AB and CD the same distance apart from A to C as from B to D?



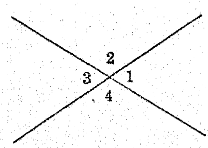


5. Which lines are continuations of what other lines?

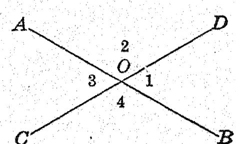
44. What is logical proof? When you establish that a statement is valid by reasoning alone and without the aid of experimentation, you have given a logical proof. Observe the difference between *experimentation* and *logical proof* in the examples given below.

45. Vertical angles. When two straight lines intersect, a pair of non-adjacent angles formed are called **vertical angles.**

Thus $\angle$s 1 and 3 are vertical angles, as are also $\angle$s 2 and 4.



46. Are vertical angles always equal? In the figure at the right, AB and CD are straight lines intersecting at O and forming angles 1, 2, 3, and 4.



Observation: $\angle 1$ seems to be equal to $\angle 3$ and $\angle 2$ seems to be equal to $\angle 4$, in this case at least.

Experiment: Measure the angles with your protractor, or test their size with your compasses. Do the vertical angles seem to be equal? Draw other figures like the one above and test. If $\angle 1 = 30°$, how large is $\angle 2$? $\angle 3$? How large if $\angle 1 = 45°$? If $\angle 1 = 60°$?

From your observation and measuring you can now conclude:

In every case that I have tested, the vertical angles *seem* to be equal.

Proof: Observe that if you know that $\angle 1 = 40°$, the number of degrees in $\angle 2$ is found by subtracting 40° from 180°. Thus $\angle 2 = 180° - 40° = 140°$. Also to find the number of degrees in $\angle 3$, you subtract the number of degrees in $\angle 2$ from 180°. Thus $\angle 3 = 180° - 140° = 40°$.

Hence, if $\angle 1 = x°$, $\angle 2 = 180° - x°$.

Since $\angle 2 = 180° - x°$, $\angle 3 = 180° - (180° - x°) = x°$.

Hence $\angle 1 = \angle 3 = x°$.

In the same way you can show that $\angle 2 = \angle 4 = 180° - x°$.

Thus you have *proved* informally:

Vertical angles are equal.

EXERCISES

1. James Wilkins is 8 years old. Every year that he can remember it has rained on July 4. Why can he not logically draw the conclusion: *It always rains on July 4?*

2. Marion has found 4-leafed clovers and 5-leafed clovers, but she has never found a 6-leafed clover. Can she correctly conclude: *Clover stems with more than 5 leaves do not exist?* Give reasons.

3. Ten thousand children were inoculated with anti-paralysis vaccine, and none of them had infantile paralysis. Which of these conclusions can be drawn? Give reasons.

a. A child inoculated with anti-paralysis vaccine will never contract infantile paralysis.

b. A child inoculated with anti-paralysis vaccine is not likely to contract infantile paralysis.

c. No valid conclusion can be drawn.

4. What conclusions, if any, can you draw from each of the following?

a. Mrs. Mailey is Bob's mother. Mrs. Bruce is Mrs. Mailey's sister.

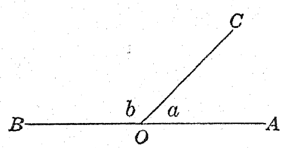
b. Last winter was the coldest winter in 10 years. This winter is colder than last winter.

5. If it rains, the ground is wet. The ground is wet. Can you conclude that it has rained? Give reasons.

6. Any pupil having home room 107 is a senior. If Mary Brown has home room 107 can you conclude that she is a senior? If Jack Wheeling has home room 103 can you conclude that he is not a senior? If Marion Jones is a pupil and has home room 107, can you conclude that she is a senior?

7. Angles a and b are supplementary. Can you conclude "AOB is a straight line"? Give reasons.

8. In the figure for Ex. 7, if AOB is a straight line, can you conclude that "$\angle$s a and b are supplementary"?



The Rules of Demonstrative Geometry

There are rules to guide you in giving a geometric proof or demonstration. In this way the study of geometry is like a game. Just as following the rules of the game help to make the game more fun, so there is pleasure in discovering geometric relations and then proving the truth of your conclusions by the rules.

47. Rules for giving a geometric proof. The Greeks were the first to state the rules which make the study of demonstrative geometry interesting. They are:

1. *The only instruments that can be used in making constructions are compasses and an unmarked straightedge.*
2. *A reason must be given for every statement made.*

3. *The only reasons that can be given are:*

- *a.* the facts that are given
- *b.* definitions
- *c.* postulates and axioms
- *d.* previously proved theorems.

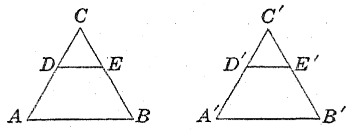
48. Undefined terms. We have not defined the terms point, line, straight line, surface, and angle because the ideas they represent are so fundamental that they cannot be defined in simpler terms.

49. We have accepted the following postulates:

1. *A straight line can be produced (drawn) to any required length.*
2. *Two straight lines cannot intersect in more than one point.*
3. *Through two given points one and only one straight line can be drawn.*
4. *The length of the line segment connecting two points is the shortest distance between them.*
5. *A circle may be drawn with any point as center and with any line segment as radius.*
6. *All radii and all diameters of the same circle or of equal circles are equal.*
7. *A geometric figure may be moved without changing its size or shape.*
8. *A line segment has one and only one point of bisection.*
9. *All right angles are equal.*
10. *An angle has one and only one bisector.*

50. An **axiom** is a statement about quantities in general which is accepted as true, without proof. You are familiar with some of the following axioms from your work in algebra.

1. *Quantities which are equal to the same quantity, or to equal quantities, are equal to each other.*



Thus, if $x = a$, and $y = a$, then $x = y$.

Also, if $AC = BC$, $A'C' = B'C'$, and $BC = B'C'$, then $AC = A'C'$.

2. *If equals are added to equals, the sums are equal.*

Thus, if $2x - 5 = 7$, then $2x = 7 + 5$.

Also, if $AD = BE$, and $DC = EC$, then $AD + DC = BE + EC$.

3. *If equals are subtracted from equals, the remainders are equal.*

Thus, if $3y + 1 = 7$, then $3y = 7 - 1$.

Also, if $AC = BC$, and $DC = EC$, then $AC - DC = BC - EC$.

4. *If equals are multiplied by equals, the products are equal.*

Thus, if $\frac{x}{3} = 2$, then $x = 6$.

Also, if $AD = \frac{1}{2} AC$, then $2 AD = AC$.

5. *If equals are divided by equals (not zero), the quotients are equal.*

Thus, if $5x = 10$, then $x = 2$.

Also, if $2 AD = AC$, $AD = \frac{1}{2} AC$.

NOTE. — When equals are divided by 2 we may say,

Halves of equals are equal.

6. *The whole of a quantity is equal to the sum of all its parts and is greater than any of its parts.*

Thus, $AC = AD + DC$. Also, AC is greater than ($>$) AD, and $AC > DC$.

7. *A quantity may be substituted for its equal in any expression.*

Thus, in $x^2 + 2x - 7$, if $x = -3$, then $(-3)^2 + 2(-3) - 7$ is the value of the expression.

Also (see figure on opposite page), in $AD + DC = BE + EC$, instead of $AD + DC$ and $BE + EC$ we can substitute AC and BC, respectively, and have $AC = BC$.

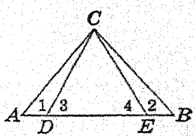
51. The following theorems have been proved informally:

1. *A right angle is half a straight angle.* (§ 28.)
2. *All straight angles are equal.* (§ 31.)
3. *Equal angles have equal complements.* (§ 39.)
4. *Equal angles have equal supplements.* (§ 40.)
5. *Vertical angles are equal.* (§ 46.)
6. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.* (§ 46, Ex. 7.)
7. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.* (§ 46, Ex. 8.)

EXERCISES

In the following exercises give as a reason either a definition or one of the postulates, axioms, or theorems in §§ 49–51.

EXAMPLE 1. Given $DC = EC$. Why is $\triangle CDE$ isosceles?



ANSWER: $\triangle CDE$ is isosceles because *An isosceles triangle is a triangle with two sides equal.* (§ 14.)

EXAMPLE 2. Given $\angle 1 = \angle 2$, $ADEB$ is a straight line. Why is $\angle 3 = \angle 4$?

ANSWER: $\measuredangle$ 1 and 3 are supplementary and also $\measuredangle$ 2 and 4 (§ 38). Hence $\angle 3 = \angle 4$ because *Equal angles have equal supplements* (§ 40).

1. Given: $3x + 7 = 19$. Why is $3x = 12$?

2. Given: $5a - 4 = 1$. Why is $5a = 5$?

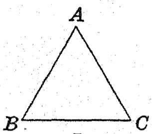
3. Given: $\frac{2}{3}x = 8$. Why is $2x = 24$?

4. Given: $7y = 28$. Why is $y = 4$?

5. Given: $\angle 1 + \angle 3 = 180°$, and $\angle 2 + \angle 4 = 180°$. Why is $\angle 1 + \angle 3 = \angle 2 + \angle 4$?

6. Given: $\angle 1 + \angle 2 = 90°$ and $\angle 2 = \angle 3$. Why is $\angle 1 + \angle 3 = 90°$?

7. Given: $\angle 1 + \angle 2 = \angle AOB$. Why is $\angle AOB > \angle 1$?



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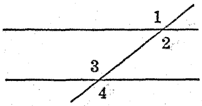
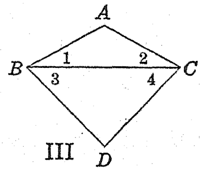


II



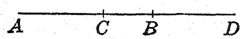
III

8. (Fig. I) Given: $AB = AC$ and $BC = AC$. Why is $AB = BC$?

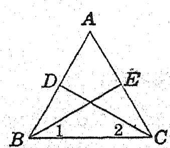
9. (Fig. II) Given: $\angle 2 = \angle 3$, $\angle 1 = \angle 2$, and $\angle 4 = \angle 3$. Why is $\angle 1 = \angle 4$?

10. (Fig. III) Given: $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Why is $\angle 1 + \angle 3 = \angle 2 + \angle 4$? Why is $\angle DBA = \angle ACD$?

11. Given: $AB = CD$. Then $AB - CB = CD - CB$ or $AC = BD$. Why?



12. Given: $\angle 1$ is $\frac{1}{2}$ $\angle CBA$ and $\angle 2$ is $\frac{1}{2}$ $\angle ACB$. Also $\angle 1 = \angle 2$. Why is $\angle CBA = \angle ACB$?



13. In the same figure, given: $BD = CE$, BD is $\frac{1}{2}$ AB and CE is $\frac{1}{2}$ AC. Why is $AB = AC$?

In Ex. 14–18 $ADEB$ is a straight line.

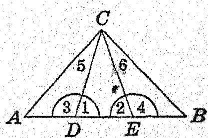
14. Given: $AC = CB$. Why is $\triangle ABC$ isosceles?

15. Given: $AD = EB$. Why is $AE = DB$?

16. Given: $AE = DB$. Why is $AD = EB$?

17. Given: $\angle 5 = \angle 6$. Why is $\angle ACE = \angle DCB$?

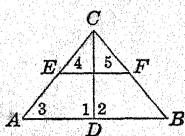
18. Given: $\angle ACE = \angle DCB$. Why is $\angle 5 = \angle 6$?



19. Given: D bisects AB. Why is $AD = DB$?

20. Given: $AC = BC$, E is the mid-point of AC, and F is the mid-point of BC. Why is $AE = BF$?

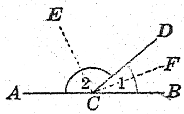
21. Given: $\angle 1 = \angle 2$. Why is $CD \perp AB$?



22. Given: $\angle 3 = \angle 4$, and $\angle 4 = \angle 5$. Why is $\angle 3 = \angle 5$?

23. Given: $AB = AC = BC$. Why is $\triangle ABC$ equilateral?

24. Given: ACB is a straight line. Why are $\angle$s 1 and 2 supplementary?



25. Given: $\angle 1 + \angle 2 = 1$ st. $\angle$. Why is ACB a straight line?

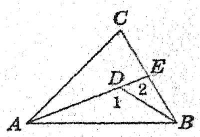
26. Given: CF bisects $\angle 1$, and CE bisects $\angle 2$. Why is $\angle BCF = \angle FCD$, and $\angle DCE = \angle ECA$?

27. Given: $\angle FCD + \angle DCE = 1$ rt. $\angle$. Why is $EC \perp CF$?

28. Given: $\angle$s 1 and 2. Why is $\angle BCA = \angle 1 + \angle 2$? (See Axiom 6.)

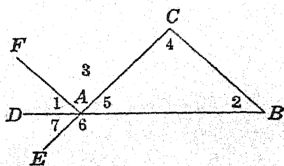
Supply reasons for the statements in the following:

29. Given: ADE is a straight line.

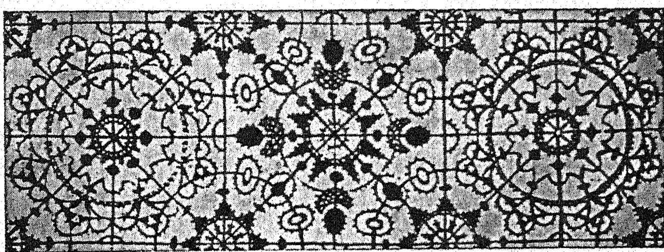


1. $BE + EC = BC$.
2. $\angle$s 1 and 2 are supplementary.
3. $AD + DB > AB$.
4. $AD + DE = AE$.
5. $AC + CE > AE$.

30. Given DAB and EAC straight lines, $\angle FAD = \angle CBA$ and $\angle CAF = \angle ACB$.



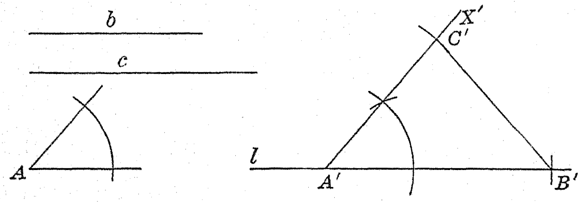
1. $\angle CAD = \angle CAF + \angle FAD$.
2. $\angle CAD > \angle FAD$.
3. $\angle 1 + \angle 3 = \angle 2 + \angle 4$.
4. $\angle CAD = \angle B + \angle C$.
5. $\angle 1 + \angle 3 + \angle 5 = \angle 5 + \angle 6$.
6. $\angle 5 + \angle 6 = \angle 6 + \angle 7$.



A PATTERN OF XVI CENTURY ITALIAN LACE

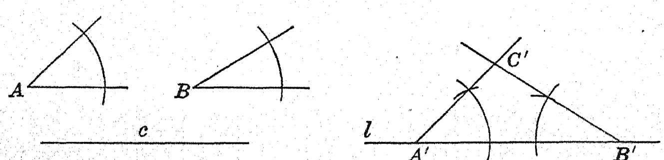
More Constructions

52. To construct a triangle when two sides and the included angle are given.



Construction: 1. From any point A' on line l take $A'B'$ equal to c. 2. At A' construct $\angle B'A'X'$ equal to $\angle A$. 3. From A' on $A'X'$, take $A'C'$ equal to b. 4. Draw $B'C'$. 5. $\triangle A'B'C'$ is the required triangle.

53. To construct a triangle when two angles and the included side are given.



Construction: 1. From any point A' on line l, take $A'B'$ equal to c. 2. At A' construct an angle equal to $\angle A$ and at B' an angle equal to $\angle B$. Call the intersection formed C'. 3. $\triangle A'B'C'$ is the required triangle.

EXERCISES

Construct the following triangles. Cut out and compare each triangle in size and shape with the ones constructed by your classmates.

1. $AB = 2$ in., $AC = 2\frac{1}{2}$ in., $\angle A = 40°$.
2. $AB = 2$ in., $AC = 2\frac{1}{2}$ in., $\angle A$ a right angle.
3. $AB = 2$ in., $AC = 2\frac{1}{2}$ in., $\angle A = 120°$.

4. $AB = AC = 3$ in., $\angle A = 80°$. What kind of triangle is this? Why?

5. $AB = 3$ in., $\angle A = 50°$, $\angle B = 65°$.

6. $AB = 2\frac{1}{2}$ in., $\angle A = 75°$, $\angle B = 110°$.

7. $AB = 2\frac{1}{2}$ in., $\angle$s A and B each 45°.

Congruent Triangles

54. Congruent (cŏn′gru-ĕnt) figures are the same in size and in shape.

If two geometric figures can be made to coincide in all their parts, they are said to be **congruent** (symbol $\cong$).

Superposition is the placing of one geometric figure on another.

Experiment. Construct triangle ABC. Make $AB = 2\frac{1}{2}$ in., $AC = 3\frac{1}{2}$ in., and angle $A = 50°$. Construct another triangle $A'B'C'$ with $A'B' = 2\frac{1}{2}$ in., $A'C' = 3\frac{1}{2}$ in., and angle $A' = 50°$. You have constructed two triangles with two sides and the included angle of one equal, respectively, to two sides and the included angle of the other. Do you think such triangles are congruent? Cut them out and test by placing triangle $A'B'C'$ on triangle ABC so that $A'B'$ coincides with its equal AB, A' falling on A and C and C' falling on the same side of AB.

Why will $A'C'$ fall along AC? (§ 23.)

Why will C' fall on C?

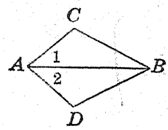
Then why will $B'C'$ coincide with BC? (Post. 3.)

We shall assume the truth of the theorem:

55. Proposition 1. *If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of another, the triangles are congruent.* (Abbreviated, *s.a.s.* = *s.a.s.*)

The following exercises depend for their proof on this theorem.

EXERCISES

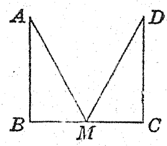


1. **Given:** $AC = AD$.
$\angle 1 = \angle 2$.

To prove: $\triangle ACB \cong \triangle ADB$.

Proof:

STATEMENTS	REASONS
1. $AB = AB$.	1. *The same line.*
2. $AC = AD$.	2. *Given.*
3. $\angle 1 = \angle 2$.	3. *Given.*
4. $\triangle ACB \cong \triangle ADB$.	4. *s.a.s. = s.a.s.*



2. **Given:** $AB \perp BC$, $DC \perp BC$.
$AB = DC$. M is the mid-point of BC.

To prove: $\triangle ABM \cong \triangle DCM$.

Proof:

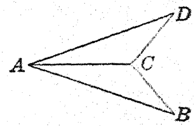
STATEMENTS	REASONS
1. $AB \perp BC$, $DC \perp BC$.	1. *Given.*
2. $\angle$s B and C are right $\angle$s.	2. *The angle formed by perpendicular lines is called a right angle.* (§ 28.)
3. $\angle B = \angle C$.	3. *All right angles are equal.* (Post. 9.)
4. M is the mid-point of BC.	4. *Given.*
5. $BM = MC$.	5. *A line segment is bisected when a point divides it into two equal parts.*
6. $AB = DC$.	6. *Given.*
7. $\triangle ABM \cong \triangle DCM$.	7. *s.a.s. = s.a.s.*

3. In the figure for Ex. 2

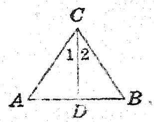
Given: $\angle BAM = \angle MDC$.
$AM = MD$.
$AB = DC$.

To prove: $\triangle ABM \cong \triangle DCM$.

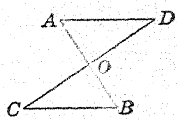
4. **Given:** $CD = CB$.
$\angle DCA = \angle ACB$.
To prove: $\triangle ACD \cong \triangle ACB$.



5. **Given:** $AC = BC$.
$\angle 1 = \angle 2$.
To prove: $\triangle ADC \cong \triangle BDC$.

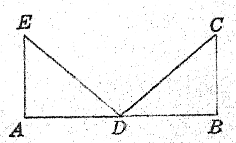


6. **Given:** AB and CD st. lines.
$AO = OB$.
$CO = OD$.
To prove: $\triangle AOD \cong \triangle COB$.

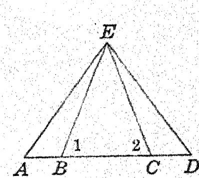


7. In the figure for Ex. 5:
Given: AB is bisected at D.
$\angle DAC = \angle CBD$.
$AC = BC$.
To prove: $\triangle ADC \cong \triangle BDC$.

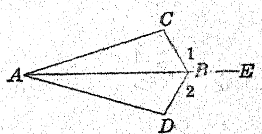
8. **Given:** $\angle$s A and B are right $\angle$s.
$AE = BC$.
$AD = DB$.
To prove: $\triangle ADE \cong \triangle BDC$.



9. **Given:** $ABCD$ is a straight line.
$\angle 1 = \angle 2$.
$BE = CE$.
$AB = CD$.
To prove: $\triangle ABE \cong \triangle DCE$.



10. **Given:** ABE a straight line.
$\angle 1 = \angle 2$.
$BC = BD$.
To prove: $\triangle ABC \cong \triangle ABD$.



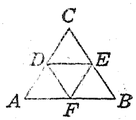
11. In the figure for Ex. 5:

Given: $CD \perp$ bis. AB.

To prove: $\triangle ADC \cong \triangle BDC$.

12. **Given:** $\triangle ABC$, with $AC = BC$.
D, E, and F mid-points of the sides.
$\angle A = \angle B$.

To prove: $\triangle ADF \cong \triangle BEF$.



***13.** In the figure for Ex. 9; if $AE = DE$, $BE = CE$ and $\angle AEB = \angle CED$, prove that $\triangle ACE$ is congruent to $\triangle BDE$.

***14.** Prove that any point on the perpendicular bisector of a segment is equidistant from the ends of the segment.

***15.** From any point P on the bisector of $\angle A$ lines PB and PC are drawn to the sides of the angle so that $AB = AC$. Prove that $PB = PC$.

56. Experiment. Construct triangle ABC. Make $AB =$ 3 in., angle $A = 50°$, and angle $B = 60°$. Construct another triangle $A'B'C'$ with $A'B' = 3$ in., angle $A' = 50°$, and angle $B' = 60°$. You have thus constructed two triangles with two angles and the included side of one equal, respectively, to two angles and the included side of the other. Do the triangles appear to be congruent? Cut them out and test by placing triangle $A'B'C'$ on triangle ABC so that $A'B'$ coincides with its equal AB, A' falling on A and C' and C falling on the same side of AB.

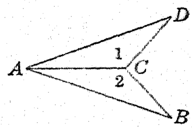
Why will $A'C'$ fall along AC? (§ 23.)
Why will $B'C'$ fall along BC? (§ 23.)
Why will points C' and C coincide? (Post. 2.)

We shall assume the truth of the theorem:

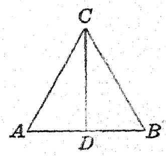
57. Proposition 2. *If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of another, the triangles are congruent.* (*a.s.a.* $=$ *a.s.a.*)

EXERCISES

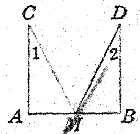
1. **Given:** AC bisects $\angle A$.
$\angle 1 = \angle 2$.
To prove: $\triangle ACD \cong \triangle ACB$.



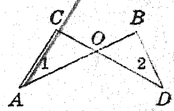
2. **Given:** $CD \perp AB$.
CD bisects $\angle C$.
To prove: $\triangle ADC \cong \triangle BDC$.



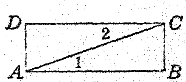
3. **Given:** $CA \perp AB$.
$DB \perp AB$.
$AC = BD$.
M is the mid-point of AB.
To prove: $\triangle ACM \cong \triangle BDM$.



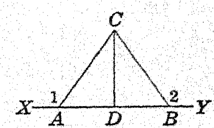
4. **Given:** AB and CD straight lines.
$AO = OD$.
$\angle 1 = \angle 2$.
To prove: $\triangle ACO \cong \triangle DBO$.



5. **Given:** $\measuredangle s$ BAD and DCB rt. $\measuredangle s$.
$\angle 1 = \angle 2$.
To prove: $\triangle ADC \cong \triangle ABC$.

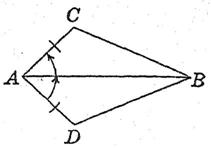


6. **Given:** $AC = BC$, XY a straight line.
$\angle 1 = \angle 2$.
$AD = DB$.
To prove: $\triangle ACD \cong \triangle BCD$.



58. Corresponding parts of congruent figures. When two figures are congruent, they can be made to coincide. The parts that match when one is superposed on the other are called **corresponding parts.** Hence: *Corresponding parts of congruent figures are equal.*

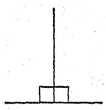
In Ex. 1 § 55 you have:

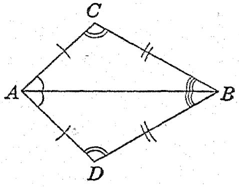


Given: $AC = AD$.
$\angle BAC = \angle DAB$.

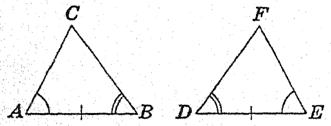
It is then proved that $\triangle ABC \cong \triangle ABD$. That is, if $\triangle ABC$ is cut out and placed on $\triangle ABD$, the two triangles can be made to coincide. You see that $\angle D$ will coincide with $\angle C$, hence $\angle D = \angle C$; BC will coincide with BD, hence $BC = BD$; and $\angle CBA$ will coincide with and equals $\angle ABD$.

Note. — It is sometimes convenient to mark the parts of a triangle which are equal, as shown, using checks on the lines and arcs for the angles. Another convenient marking is shown for right angles.

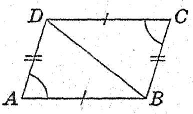




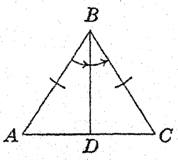
EXERCISES



I



II

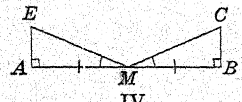


III

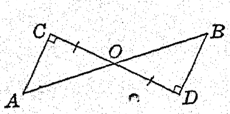
1. In I, $AB = DE$, $\angle B = \angle D$, $\angle A = \angle E$, and $\triangle ABC \cong \triangle DEF$. (Why?) Name three other pairs of corresponding parts.

2. In II, $AB = DC$, $\angle A = \angle C$, $AD = BC$, and $\triangle ABD \cong \triangle BCD$. (Why?) Name the other corresponding parts.

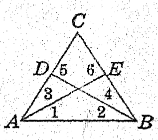
3. In III, $AB = BC$, $\angle ABD = \angle DBC$. Why are the triangles congruent? Name the other corresponding parts.



IV



V



VI

4. In IV, M is the mid-point of AB, EA and $CB \perp AB$, and $\angle EMA = \angle BMC$. Why are the triangles congruent? Name the other corresponding parts.

5. In V, AB and CD are straight lines, $OC = OD$, AC and $DB \perp CD$. Why are the triangles congruent? Name the corresponding parts.

Ex. 6–8 refer to Fig. VI.

6. How many triangles can you name? How many seem congruent?

7. If $AE = BD$ and $\angle 1 = \angle 2$, why is $\triangle ABE \cong \triangle ABD$? Name the corresponding parts.

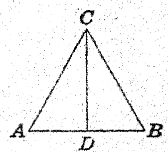
*8. If $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, why is $\triangle ABE \cong \triangle ABD$? Name the corresponding parts.

59. Two of the powerful tools you will use in solving originals are:

I. *To prove that two line segments are equal show that they are corresponding sides of congruent triangles.*

II. *To prove that two angles are equal show that they are corresponding angles of congruent triangles.*

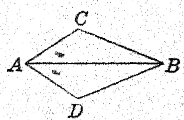
EXERCISES



Ex. 1, 2

1. **Given:** $AC = BC$, CD bisects $\angle ACB$.
To prove: $\angle A = \angle B$.

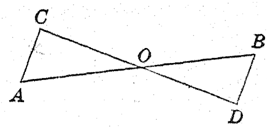
2. In the same figure:
Given: $CD \perp AB$, $\angle ACD = \angle DCB$.
To prove: $\angle A = \angle B$.



Ex. 3, 4

3. **Given:** $\angle CBA = \angle ABD$, $BC = BD$.
To prove: $\angle C = \angle D$.

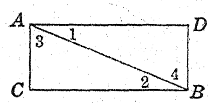
4. In the same figure:
Given: AB bisects $\angle CBD$ and $\angle DAC$.
To prove: $CB = DB$.



5. **Given:** O bisects CD, AC and $BD \perp CD$.
AB and CD are straight lines.
To prove: $\angle A = \angle B$.

6. In the same figure:
Given: Lines AB and CD are bisected at O.
To prove: $AC = BD$.

7. **Given:** $\angle 1 = \angle 2$, $AD = BC$.
To prove: $BD = AC$.

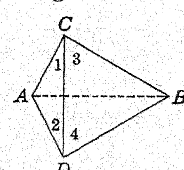


8. In the same figure:
Given: $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.
To prove: $\angle D = \angle C$.

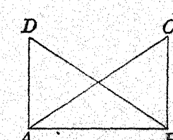
---

9. In the figure for Ex. 1:
Given: CD is the perpendicular bisector of AB.
To prove: $\triangle ABC$ is isosceles.

10. In the figure for Ex. 7:
Given: $\angle 1 = \angle 2$, $\angle CAD = \angle DBC$, $\angle C$ is a right $\angle$.
To prove: $BD \perp AD$.



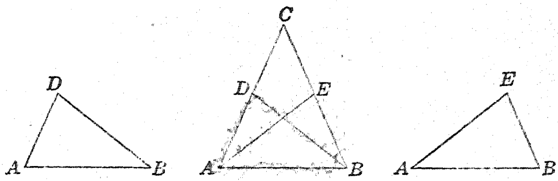
11. **Given:** $AC = AD$, $BC = BD$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.
To prove: AB bisects $\angle CBD$.



*12. **Given:** $AD = BC$, AD and $BC \perp AB$.
To prove: $\angle ADB = \angle ACB$.

*13. **Prove:** Two adjacent angles whose bisectors form an angle of $45°$ are complementary.

OVERLAPPING TRIANGLES



60. When triangles to be proved congruent overlap you can imagine that they are moved apart, as in the illustration. Thus:

Given: $\triangle ABC$, $\angle BAC = \angle CBA$, $AD = BE$.

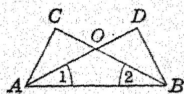
To prove: $AE = BD$.

Proof:

STATEMENTS	REASONS
1. $\angle BAC = \angle CBA$, $AD = BE$.	1. *Given.*
2. $AB = AB$.	2. *The same line.*
3. $\triangle DAB \cong \triangle EBA$.	3. *s.a.s. = s.a.s.*
4. $AE = BD$.	4. *Corresponding parts of congruent triangles are equal.*

EXERCISES

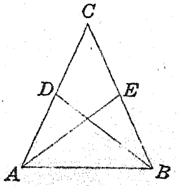
If necessary, imagine the overlapping triangles removed as in the example above.



1. **Given:** $\angle BAC = \angle DBA$.
   $\angle 1 = \angle 2$.
   To prove: $\angle C = \angle D$.
2. **Given:** $AD = BC$, $\angle 1 = \angle 2$.
   To prove: $AC = BD$.

3. **Given:** $\angle BAC = \angle CBA$, $\angle BAE = \angle DBA$.
To prove: $AE = BD$.

4. **Given:** AE bisects $\angle BAC$ and BD bisects $\angle CBA$, $\angle BAC = \angle CBA$.
To prove: $\angle ADB = \angle AEB$.

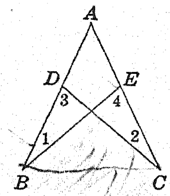


5. In Ex. 3, what further steps are necessary to prove $\angle BDC = \angle CEA$?

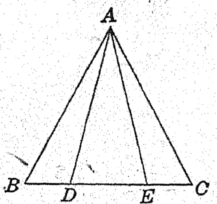
6. In the same figure:
Given: $AC = BC$, $DC = EC$.
To prove: $AE = BD$.

7. In the same figure:
Given: $AC = BC$, D bisects AC and E bisects BC.
To prove: $\angle CEA = \angle BDC$.

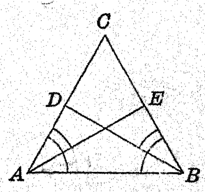
8. In the same figure:
Given: $AE = BD$, $DC = EC$, $\angle ADB = \angle AEB$.
To prove: $AC = BC$.



9. **Given:** $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $BE = CD$.
To prove: $AB = AC$.

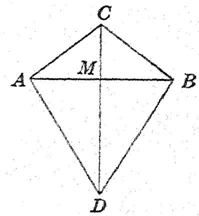


10. **Given:** $BE = DC$, $\angle B = \angle C$, $\angle AEB = \angle CDA$.
To prove: $\triangle ADE$ is isosceles.



*11. **Given:** $\angle BAC = \angle CBA$, AE bisects $\angle BAC$, BD bisects $\angle CBA$.
To prove: $AC = BC$.

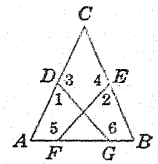
HINT. — First prove $\triangle ABD \cong \triangle ABE$. Then prove $\triangle AEC \cong \triangle BDC$.



***12. Given:** $AC = BC$, $AD = BD$, $\angle DAC = \angle CBD$.

To prove: $CD \perp$ bis. AB.

HINT. — First prove $\triangle ACD \cong \triangle BCD$. Then prove $\triangle ACM \cong \triangle BCM$.



***13. Given:** $AC = BC$.
D bisects AC. E bisects BC.
$\angle 3 = \angle 4$. $\angle A = \angle B$.

To prove: $AG = FB$.

***14. Given:** $AF = GB$.
$\angle A = \angle B$. $\angle 5 = \angle 6$.

To prove: $AD = BE$.

***15. Prove:** The bisectors of two vertical angles form a straight line.

61. Summary of the work of Unit One. In this unit you have been introduced to the study of plane geometry. You have studied:

I. ***Definitions of geometric terms.***

1. *The most important of these are:*

line segment	*equilateral*	*acute angle*
circle	*postulate*	*right, acute, and obtuse triangles*
radius	*bisected*	*equiangular triangle*
diameter	*mid-point*	*complementary*
chord	*adjacent angles*	*supplementary*
arc	*perpendicular*	*vertical angles*
polygon	*straight angle*	*congruent*
triangle	*right angle*	
isosceles		

2. *Terms undefined are point, line, straight line, surface, and angle.*

II. ***Constructions.***

1. *To copy a line segment.*
2. *To construct a triangle when three sides are given.*
3. *To bisect a line segment.*
4. *To copy an angle.*
5. *To bisect an angle.*
6. *To construct the perpendicular bisector of a line segment.*
7. *To construct a perpendicular to a line at a point on the line.*
8. *To construct a triangle when two sides and the included angle are given.*
9. *To construct a triangle when two angles and the included side are given.*

III. ***Postulates and axioms.*** *The postulates are listed in* § 49; *the axioms in* § 50.

IV. ***Theorems informally proved.***

1. *All straight angles are equal.*
2. *A right angle is half a straight angle.*
3. *Equal angles have equal complements.*
4. *Equal angles have equal supplements.*
5. *Vertical angles are equal.*
6. *If two adjacent angles are supplementary, their exterior sides lie in a straight line.*
7. *If two adjacent angles have their exterior sides in a straight line, they are supplementary.*
8. *If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of another, the triangles are congruent.*

9. *If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of another, the triangles are congruent.*

REVIEW OF UNIT ONE

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the tests. The references given are those most closely related to the exercise.

I

Supply the missing words in the following:

1. Complementary angles are two angles whose sum is ——. § 38.

2. The sum of two supplementary angles is ——. § 38.

3. If two lines intersect so as to make the adjacent angles equal, they are said to be ——. § 26.

4. An angle whose sides extend in opposite directions from the vertex and form a straight line is called a —— angle. § 27.

5. A triangle having one obtuse angle is called an —— triangle. § 30.

6. If the initial side of an angle revolves completely around to take its original position, it revolves through an angle of —— degrees. § 34.

7. Any part of a circle is called an ——. § 10.

8. If two adjacent angles have their exterior sides in a straight line, they are ——. § 38.

9. The equal parts in two congruent triangles are called —— parts. § 58.

10. Reasons that may be used in the proof of a theorem are the parts given, ——, ——, ——, and previously proved ——. § 47.

II

Complete the following:

1. Equal angles have .... §§ 39, 40.

2. Quantities equal to the same or .... § 50, 1.

3. If two adjacent angles are supplementary, their exterior sides .... § 51, 6.

4. A circle may always be drawn with .... § 49, 5.

5. Through two given points .... § 49, 3.

6. All radii and all diameters of the same or .... § 49, 6.

7. A right angle is half .... § 51, 1.

8. A quantity may be substituted .... § 50, 7.

9. An angle has one and only one .... § 49, 10.

10. A geometric figure may be .... § 49, 7.

III

In the following give reasons for your answers:

1. Does a straight line have a definite length? § 9.

2. Is the sum of all the angles on one side of a straight line 180°? § 34.

3. Is a triangle a polygon? § 13.

4. Is it true that an equilateral triangle is isosceles? § 14.

5. Are angles that have the same vertex and a common side always adjacent? § 25.

6. Is a triangle having one acute angle always called an acute triangle? § 30.

7. Can either side of an angle be taken as the initial side? §23.

8. Is it true that a straight line is a straight angle? § 27.

9. Is any angle greater than a right angle called an obtuse angle? § 29.

10. Are the bisectors of two supplementary adjacent angles perpendicular? Why? §§ 38, 26.

IV

In Ex. 1–7, make the constructions, leaving all arcs:

1. Bisect a line segment. § 21.

2. Copy an angle. § 24.

3. Construct the perpendicular bisector of a line segment. § 36.

4. Construct a triangle having three given segments as sides. § 17.

5. Construct the bisector of a given angle. § 33.

6. Construct a triangle given two sides and the included angle. § 52.

7. Construct a triangle given two angles and the included side. § 53.

8. What is the complement of $34° 20'$? Of $x°$? § 38.

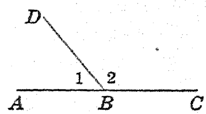
9. What is the supplement of $20° 14'$? Of $x°$? § 38.

See if you can complete the proof of the following:

10. **Given:** Adj. ∠s 1 and 2.

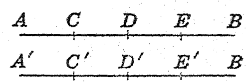
$\angle 1 = 50°, \angle 2 = 130°.$

To prove: ABC is a straight line. § 51, 6.



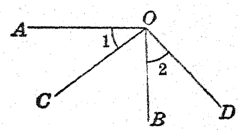
11. **Given:** $AB = A'B'$, points C, D, and E divide AB into four equal parts and points C', D', and E' divide $A'B'$ into four equal parts.

To prove: $CD = E'B'$. § 50, 5.



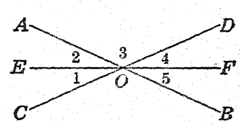
12. **Given:** $AO \perp OB$, $\angle 1 = \angle 2$.

To prove: $CO \perp OD$. § 50, 2.



13. **Given:** AB and CD straight lines intersecting at O, EO bisecting $\angle AOC$ and FO bisecting $\angle BOD$.

To prove: FOE a straight line. §§ 46, 27.



14. (Use figure for Ex. 13.)

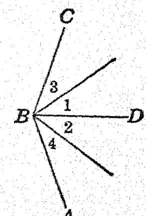
Given: AB and CD straight lines intersecting at O, forming vertical angles AOC and BOD; EO bisects $\angle AOC$ and is produced to F.

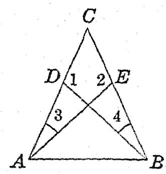
To prove: OF bisects $\angle BOD$. § 46.

15. Two straight lines intersect at a point O, forming four angles. If one of the angles is 50°, how large is each of the others? §§ 46, 34.

16. **Given:** BD bisects $\angle ABC$, $\angle 1 = \angle 2$.

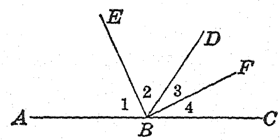
To prove: $\angle 3 = \angle 4$. § 50, 3.





*17. **Given:** $AD = BE, AE = BD, \angle 1 = \angle 2.$
To prove: $\angle BAD = \angle EBA.$ §§ 40, 55.

*18. In the same figure:
Given: $DC = CE, \angle 1 = \angle 2.$
To prove: $\angle 3 = \angle 4.$ § 57.



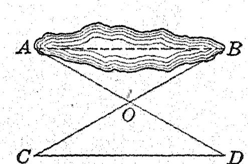
*19. **Given:** ABC is a straight line. BE, BD, and BF are drawn so that $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.
To prove: $BF \perp BE.$ §§ 26, 38.

PRACTICAL APPLICATIONS

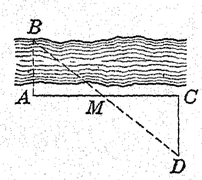
(Optional)

Applications of geometry. *Throughout the text you will find the practical applications of geometry marked optional. This is not because they are not important. Work as many of them as time will permit.*

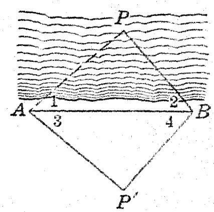
1. In order to find the distance from A to B across a lake, at camp, some boy scouts first measured off a straight line AO and extended it to D so that $AO = OD$. Second, they measured the distance BO and extended BO to C so that $BO = OC$. Then they measured the distance from C to D. Prove that AB equals the measured distance CD.



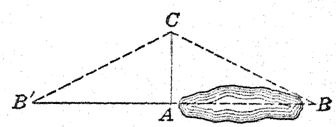
2. The distance AB across a stream may be found as follows: Measure AC at right angles to AB. Locate a point M halfway between A and C. Measure CD at right angles to AC, to a point D in line with B and M. Prove that AB equals CD.



3. It is said that Thales (about 600 B.C.), determined the distance AP of a ship from shore by means of the congruence of triangles. He measured $\angle 1$ and $\angle 2$. Explain how he was then able to mark off a distance on the shore equal to AP and thus find the distance AP.

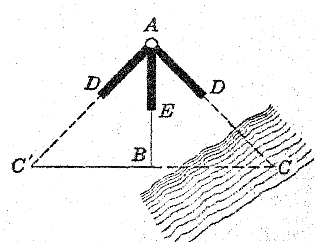


4. One of Napoleon's young lieutenants is said to have earned promotion by quickly determining the distance across a river as follows: He stood at A and looked at B on the opposite side of the river, lowering his head until his hat brim fell in the line of sight. Then, without raising or lowering his head, he turned about, and observed the point B' at which the line of sight struck the ground. He quickly paced the distance from A to B' and told Napoleon the result.

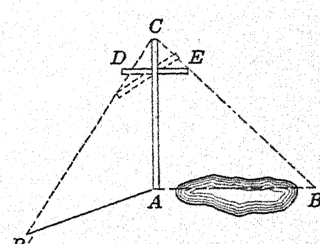


NOTE. — In the diagram, C represents the position of the eyes. Prove that $AB = AB'$.

5. Thales (see Ex. 3) is said to have made an instrument for determining the distance BC of a ship from shore. It consisted of two rods AD and AE, hinged together at A. Rod AE was held vertically over point B, while rod AD was pointed toward C. Then, without changing $\angle DAE$, the instrument was revolved about AE, and point C' noted on the ground at which the arm AD was directed. BC' was then measured. Prove that $BC = BC'$.

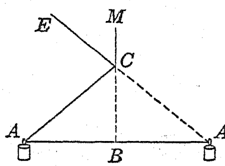


6. An instrument used as late as the sixteenth century for finding the distance from A to an inaccessible point B was called a cross-staff. It consisted of a vertical staff AC to which was attached a horizontal



cross bar DE that could be moved up or down on the staff. Sighting from C to B, DE was lowered or raised until C, E, and B were in a straight line. Then the instrument was revolved about CA and the point B' at which the line of sight CD met the ground was marked. The distance AB' was then measured. Prove that $AB = AB'$.

7. Every one is familiar with the fact that if an object is placed before a plane mirror, its image appears to be as far behind the mirror as the object is in front. M is an edge view of a mirror. Light from an object at A strikes the mirror at C and is reflected to the eye at E. The mind projects the line EC through the mirror to A', forming the image at A'. It is known from science that $\angle MCE = \angle ACB$ and that $CB \perp AA'$. Prove that $AB = A'B$.



PRACTICE TESTS ON UNIT ONE

These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TEST ONE

Numerical Exercises

For the following exercises, see § 34 *and* § 38.

1. What is the complement of 23° 45′?

2. What is the supplement of 70°?

3. How many degrees in a straight angle?

4. The diameter of a circle is 18 in. How long is the radius? § 10.

5. How many degrees in the smaller angle between the hands of a clock at 4 o'clock?

6. Two straight lines intersect and form four angles. If one of the angles is 17°, how large is each of the others? § 51, 5.

7. What is the sum of all the angles about a point?

8. An angle contains x degrees. How many degrees in its complement?

TEST TWO

Matching Exercises

In one column brief descriptions of the terms in the other column are given. Match them correctly.

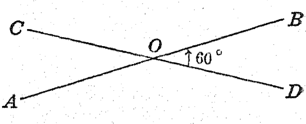
I. Vertex. § 12.	1. Connected parts of straight lines.
II. Right triangle. § 30.	2. Two angles which have a common vertex and a common side between them.
III. Complementary angles. § 38.	3. An angle of 180°.
IV. Postulate. § 18.	4. The point where two sides of a polygon meet.
V. Adjacent angles. § 25.	5. A proposition to be proved.
VI. Supplementary angles. § 38.	6. A definite part of a straight line.
VII. Equiangular. § 30.	7. Two angles whose sum is 180°.
VIII. Corresponding parts. § 58.	8. A line no part of which is straight.
IX. Theorem. § 41.	9. A triangle having one right angle.
X. Obtuse angle. § 29.	10. The opening between two lines which meet.
XI. Broken line. § 9.	11. Two angles whose sum is 90°.
XII. Angle. § 23.	12. An angle more than 90° and less than 180°.
XIII. Curved line. § 9.	13. Having all angles equal.
XIV. Line segment. § 9.	14. An assumption made without proof in geometry.
XV. Straight angle. § 27.	15. Parts similarly placed.

TEST THREE

Supplying Reasons

Supply definitions, axioms, postulates, or theorems as reasons for the following statements.

1. An arc is less than a circle. § 50, 6.
2. If $x = y$, then $x + \sqrt{a} = \sqrt{a} + y$. § 50, 2.
3. The sum of two sides of a triangle is greater than the third side. § 49, 4.
4. If $a + \frac{5}{6}b = c + \frac{1}{6}d$, then $6a + 5b = 6c + d$. § 50, 4.
5. If angles A and D are right angles, they can be made to coincide. § 49, 9.
6. If $\angle A = \angle B$ and $\angle A$ is 70°, the supplement of $\angle B$ is 110°. § 51, 4.
7. If $AB = MN$ and $MN = XY$, then $AB = XY$. § 50, 1.
8. In triangle ABC, if $AB = AC$, the triangle is isosceles. § 14.
9. Two straight lines AB and CD intersect at O. If angle $DOB = 60°$, angle COA is 60° also. § 51, 5.
10. Angle DOC is greater than angle DOB. § 50, 6.



TEST FOUR

Completing Statements

Complete the following statements.

1. An acute triangle is a triangle having —— acute angles. § 30.
2. Adjacent angles are angles having a common —— and a —— side —— them. § 25.
3. The size of an angle does —— depend on the —— of its sides, but on the amount of —— of one of the sides. § 23.
4. The equal sides of an isosceles triangle are sometimes called the ——. § 14.

5. A straight angle is an angle whose sides extend —— —— —— from the vertex and form a —— ——. § 27.

6. Two lines which meet and form —— —— angles are perpendicular. § 26.

7. A —— is an instrument used to measure angles. § 34.

8. An —— triangle has one obtuse angle. § 30.

9. If two triangles are congruent, —— angles are equal and corresponding sides are ——. § 58.

10. —— are used to transfer segments in geometry. § 15.

TEST FIVE

Multiple Choice Statements

From the expressions printed in italics select that one which best completes the statement.

1. If the sum of two angles is 180°, the angles are *each 90°*, *adjacent*, *supplementary*. § 38.

2. The vertex of a triangle is *the intersection of two sides*, *an acute angle*. § 12.

3. A polygon has *three or more*, *more than three*, *more than four* sides. § 12.

4. If two triangles have the angles of one equal respectively to the angles of the other, the triangles *may not be congruent*, *are congruent*, *are equiangular*. §§ 30, 55, 57.

5. Two complementary angles *are each 45°*, *are adjacent*, *have their sum 90°*. § 38.

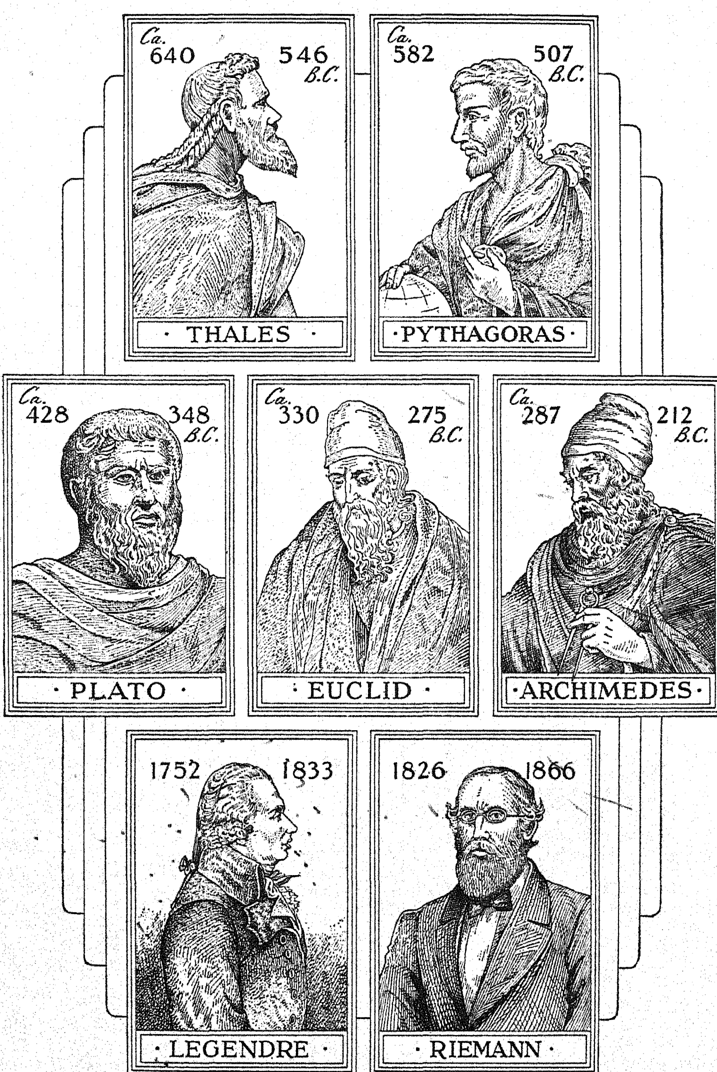
6. The sum of all the angles about a point is *360°*, *180°*, *two right angles*. § 34.

7. The bisectors of a pair of vertical angles *are equal*, *lie in a straight line*, *bisect the other pair of vertical angles*. Page 59, Ex. 13.

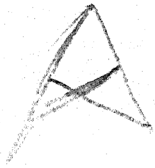
8. A geometric statement assumed to be true without proof is called a *proposition*, *problem*, *postulate*. § 18.

9. An acute triangle is a triangle having *only one acute angle*, *only two acute angles*, *three acute angles*. § 30.

10. The sum of any two sides of a triangle is *greater than*, *equal to*, *less than* the third side. § 49, 4.



Eminent Names in the History of Geometry



UNIT TWO

FORMAL DEMONSTRATION; ANALYSIS; CONSTRUCTIONS; INDIRECT PROOF

In the first unit you have seen what is meant by a geometric proof and have acquired a knowledge of definitions, axioms, postulates, and theorems which you can use in giving such proofs. In this unit you will learn more about such proofs and how you can discover them.

62. A **proposition** is a statement of a geometric truth. A **theorem** is a proposition to be proved.

The **proof** is the argument which shows that the proposition is true.

63. Formal demonstration. When you prove theorems in geometry, you must follow a very definite form. Such a proof is called a **demonstration.** The steps in writing a demonstration follow:

1. *Write the theorem.*

2. *Draw and letter a figure to represent the geometric ideas given in the theorem.*

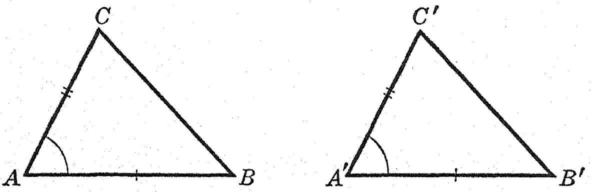
3. *Separate the theorem into its two parts, what is given and what is to be proved, using the letters from your figure to express the relations.*

4. *Any additional lines it is necessary to draw in your figure to aid in the proof should be dotted lines.*

5. *Number your statements and opposite each write a reason.*

Proposition 1. Theorem

64. B. *If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of another, the triangles are congruent.* (*s.a.s.* = *s.a.s.*)



Given: $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$.

To prove: $\triangle A'B'C' \cong \triangle ABC$.

Plan: Use the method of superposition.

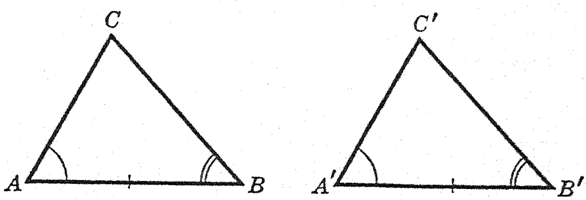
Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with its equal AB, A' falling on A, with C' and C on the same side of AB.	1. *A geometric figure may be moved without changing its size or shape.* *Given* $AB = A'B'$.
2. $A'C'$ will take the direction of AC.	2. *Given* $\angle A = \angle A'$.
3. C' will fall on C.	3. *Given* $AC = A'C'$.
4. $B'C'$ will coincide with BC.	4. *Through two points one and only one straight line can be drawn.*
5. $\triangle A'B'C' \cong \triangle ABC$.	5. *If two figures can be made to coincide, they are congruent.*

This proof and that given on page 69 are for the theorems we assumed in §§ 55 and 57.

PROPOSITION 2. THEOREM

65. B. *If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of another, the triangles are congruent.* (*a.s.a.* = *a.s.a.*)



Given: $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $\angle A = \angle A'$, $\angle B = \angle B'$.

To prove: $\triangle A'B'C' \cong \triangle ABC$.

Plan: Superposition.

Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with its equal AB, point A' falling on point A, with C' and C on the same side of AB.	1. *A geometric figure may be moved without changing its size or shape. Given* $AB = A'B'$.
2. $A'C'$ will take the direction AC.	2. *Given* $\angle A = \angle A'$.
3. $B'C'$ will take the direction BC.	3. *Given* $\angle B = \angle B'$.
4. C' will fall on C.	4. *Two straight lines cannot intersect in more than one point.*
5. $\triangle A'B'C' \cong \triangle ABC$.	5. *If two figures can be made to coincide, they are congruent.*

66. How to study geometry. The following rules will help you to study your geometry.

1. *Have a regular time and regular place to study, free from distractions.*
2. *Have at hand pencil, paper, ruler, and compasses.*
3. *Read the theorem over several times until you are familiar with it, making certain that you know the exact definition of each term used.*
4. *Draw your figure carefully, and not too small. Make all construction lines required with your straightedge and compasses.*
5. *Make the figure general; that is, if you are drawing "a triangle," do not draw a right triangle nor an isosceles triangle.*
6. *Write the hypothesis (given) and the conclusion (to prove) in terms of the figure.*
7. *State your plan of proof.*
8. *See if you can write out the proof without looking at the formal proof in the book. Remember to give a reason for each statement. Number the statements and reasons alike. (Reasons must never be given by number. They must be written out as in* §§ 64 *and* 65.)
9. *If you cannot write the proof without aid, use the suggestions given in the plan. If you still cannot, read the proof given. See if you can supply the reasons without looking up the section numbers.*

Refer to these rules frequently and try to develop power to prove theorems without aid.

67. Parts of a theorem. Every theorem consists of two parts: that which is given, and that which is to be

proved. The facts given are called the **hypothesis,** and that which you are to prove is called the **conclusion.**

A theorem can always be written with two clauses: the hypothesis beginning with "if," and the conclusion beginning with "then." For example, the theorem *Vertical angles are equal* can be written: If *two straight lines intersect,* then *the vertical angles formed are equal.* The clause, *If two straight lines intersect,* is the hypothesis and contains that which is given in the theorem. The clause, *then the vertical angles formed are equal,* is the conclusion and contains that which is to be proved.

In the following exercises write out the theorems, putting one line under the hypothesis and two lines under the conclusion. Then draw and letter a figure and express in terms of the figure what is given and what is to be proved. Do not attempt to write the proofs.

EXERCISES

1. If two adjacent angles are supplementary, then the exterior sides form a straight line.

2. If a triangle is equiangular, then it is equilateral.

3. If two adjacent angles are complementary, then their bisectors form an angle of 45°.

4. If two adjacent angles are supplementary, then their bisectors are perpendicular.

5. If a triangle is isosceles, then the angles opposite the equal sides are equal.

6. If a point is on the perpendicular bisector of a line segment, then it is equidistant from the ends of the segment.

7. If a line bisects one of two vertical angles, then it bisects the other.

In Ex. 8–10, first write in the "if — then" form:

8. Two adjacent angles whose bisectors form an angle of 45° are complementary.

9. Lines joining the extremities of two line segments which bisect each other form two pairs of congruent triangles.

10. The angles of an equilateral triangle are equal.

*11. Assuming the truth of the other statements, tell which of the following conclusions are valid.

Ruth will not go if it rains.

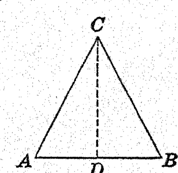
1. It rained. *Conclusion:* Ruth did not go.
2. It did not rain. *Conclusion:* Ruth went.
3. Ruth went. *Conclusion:* It did not rain.
4. Ruth did not go. *Conclusion:* It rained.

68. Auxiliary lines. To prove that two segments or that two angles are equal, it is sometimes necessary to draw auxiliary lines in order to form congruent triangles. This we shall call

Tool III. *If the line segments or angles are not parts of congruent triangles, try to make them so by drawing auxiliary lines.* In drawing such lines, always be sure not to impose too many conditions on them. You cannot say, "Draw a straight line through three points," for it will be impossible to do so unless the points lie in a straight line. Nor can you say, "Draw a line through C, bisecting angle C and perpendicular to AB."

No more than two conditions can be imposed on a line.

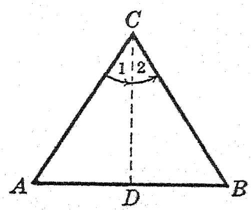
If $\triangle ABC$ is isosceles, and you wish to prove $\angle A = \angle B$, can you draw an auxiliary line to form congruent triangles? The two triangles must have either a.s.a. = a.s.a. or s.a.s. = s.a.s. Would a line from $C \perp AB$ do? Why? What about a line from C bisecting $\angle C$? Complete the proof of the theorem:



If a triangle is isosceles, the angles opposite the equal sides are equal.

PROPOSITION 3. THEOREM

69. *If a triangle is isosceles, the angles opposite the equal sides are equal.*



Given: $\triangle ABC$ with $AC = BC$.

To prove: $\angle A = \angle B$.

Plan: Suppose CD bisects $\angle C$, then prove $\triangle ADC \cong \triangle BDC$.

Proof:

STATEMENTS	REASONS
1. Suppose CD bisects $\angle C$.	1. *An angle has a bisector.*
2. $AC = BC$.	2. *Given.*
3. $\angle 1 = \angle 2$.	3. *A bisector divides an angle into two equal parts.*
4. $CD = CD$.	4. *The same line.*
5. $\triangle ADC \cong \triangle BDC$.	5. *s.a.s. = s.a.s.*
6. $\angle A = \angle B$.	6. *Corresponding parts of congruent figures are equal.*

70. A **corollary** is a theorem which is easily proved.

71. COROLLARY. *An equilateral triangle is equiangular.*

HINT. — If $AC = BC = AB$ in the figure in § 69, why is $\angle A = \angle B = \angle C$? (Apply § 69 twice.) Write out a formal proof.

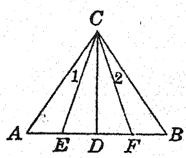
72. An **altitude** of a triangle is a line from any vertex perpendicular to the opposite side, prolonged if necessary. Every triangle has three altitudes.

73. A **median** of a triangle is a line drawn from any vertex to the mid-point of the opposite side. Every triangle has three medians.

EXERCISES

In Ex. 1–4, AEDFB is a straight line.

1. If $AC = BC$ and $AE = BF$, prove that $CE = CF$.



2. In the same figure, if $AC = BC$, $AE = BF$, and $ED = DF$, prove that $CD \perp AB$.

3. In the same figure:

Given: $AC = BC$.
$\angle 1 = \angle 2$.

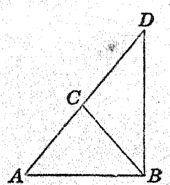
To prove: $\triangle CEF$ isosceles.

4. In the same figure:

Given: $CE = CF$, $AE = BF$.

To prove: $\triangle ABC$ isosceles.

5. In the figure, $AC = BC$ and AC is produced to D, so that $AC = CD$. Prove that $\angle A + \angle D = \angle ABD$.



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6. The bisector of the vertex angle of an isosceles triangle is also the perpendicular bisector of the base.

7. If the bisector of the angle at a vertex of a triangle is perpendicular to the opposite side, it bisects that side.

8. In any isosceles triangle the bisector of the vertex angle, the median to the base, the altitude to the base, and the perpendicular bisector of the base all coincide.

9. Prove that the medians drawn to the equal sides of an isosceles triangle are equal.

10. Prove that any two medians of an equilateral triangle are equal.

11. Prove that the bisectors of the base angles of an isosceles triangle form with the base an isosceles triangle.

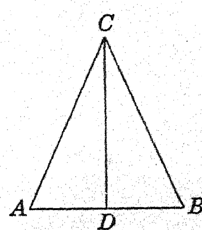
74. The meaning of converse. If you interchange one of the facts given in the hypothesis of a theorem with one of those given in the conclusion, the resulting theorem is a **converse** of the original one. Thus we have the theorem, "*If the bisector of an angle of a triangle is perpendicular to the opposite side, then the triangle is isosceles.*" A converse is "*If a triangle is isosceles, then the bisector of the vertex angle is perpendicular to the opposite side.*"

The statements in terms of a figure are:

a. THEOREM

Given: $\triangle ABC$
CD bisects $\angle ACB$
$CD \perp AB$

To prove: $AC = BC$



b. CONVERSE THEOREM

Given: $\triangle ABC$
CD bisects $\angle ACB$
$AC = BC$

To prove: $CD \perp AB$.

Both the theorem and its converse can be proved true by proving that $\triangle ADC \cong \triangle BDC$.

Another converse of *a* can be formed by interchanging another part of the hypothesis with the conclusion.

c. **Given:** $\triangle ABC$
$CD \perp AB$
$AC = BC$

To prove: CD bisects $\angle ACB$.

While this converse is also true you cannot prove it at the present time. Why not? Notice that in the hypotheses of a, b, and c, $\triangle ABC$ is a common element.

EXERCISES

1. Give the converse of each of the following statements. Tell whether the converse is true or is not true.

a. If a figure is a triangle, it has three sides.
b. If a triangle is equilateral, it is isosceles.
c. If it rains, the ground is wet.
d. If two angles are right angles, they are equal.
e. If a man lives in San Francisco, he lives in California.

2. Write a converse for each of the theorems in Ex. 1–4, § 67.

3. Write a converse for Proposition 3. Give the theorem and its converse in terms of the figure. Can you prove the converse as you did the theorem by letting CD be the bisector of angle ACB? Do triangles ADC and BDC have *s.a.s.* = *s.a.s.* or *a.s.a.* = *a.s.a.*?

In Ex. 4 and 5 give two converses and tell if they are true.

4. Given: All seniors have home room 214. Jean is a senior.

Conclusion: Jean has home room 214.

5. Given: Frank and John are brothers. Madeline is Frank's cousin.

Conclusion: Madeline is John's cousin.

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6. In terms of the figures give the converse or converses of Ex. 1–5 on pages 51–52. Tell which of them you can prove true.

7. Are the following conclusions true?

I. **Given:** Everyone in my class plays a musical instrument. Mary is in my class.

Conclusion: Mary plays a musical instrument.

II. **Given:** Everyone in my class plays a musical instrument. Mary plays a musical instrument.

Conclusion: Mary is in my class.

III. **Given:** Mary is in my class. Mary plays a musical instrument.

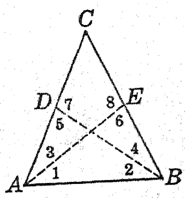
Conclusion: Everyone in my class plays a musical instrument.

Test the validity of the conclusions in Ex. 8 and 9.

8. If it stays cold, Stan will go skating. It did stay cold, therefore he went skating.

9. If it snows John will put chains on the car. He did put chains on the car, therefore it snowed.

75. We shall next prove the converse of the theorem in § 69: *If two angles of a triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.* You saw in Ex. 4 that you cannot prove the converse as you did Prop. 3.



In the figure at the right, if $\angle BAC = \angle CBA$, DB bisects $\angle CBA$ and EA bisects $\angle BAC$, why is $\triangle ABD \cong \triangle ABE$? Then $AE = BD$ and $\angle 5 = \angle 6$. Why? If $\angle 5 = \angle 6$, why does $\angle 7 = \angle 8$?

Using the facts $AE = BD$, $\angle 7 = \angle 8$, $\angle 3 = \angle 4$, why is $\triangle AEC \cong \triangle BDC$? Why is $AC = BC$? (See Ex. 11 page 54.)

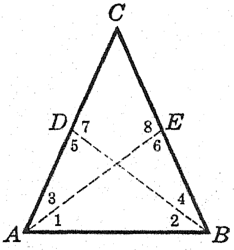
Abraham Lincoln, while president, gave an interview[1] to a reporter telling how he prepared to practice law. He related that he asked himself, "What do I do when I demonstrate more than when I reason and prove?" He searched the dictionaries and found that demonstrate meant "proof beyond possibility of doubt." He said that meant nothing more to him than "blue to a blind man."

"So," he said in the interview, "I left my situation in Springfield, went home to my father's house, and stayed there until I could give any proposition in the six books of Euclid at sight. I then found out what 'demonstrate' means, and went back to my law studies."

[1] F. B. Carpenter, *Six Months in the White House*, pp. 308–316.

PROPOSITION 4. THEOREM

76. *If two angles of a triangle are equal, the sides opposite these angles are equal.*



Given: $\triangle ABC$, $\angle A = \angle B$.

To prove: $AC = BC$.

Plan:

1. Let AE bisect $\angle A$ and BD bisect $\angle B$.
2. Prove $\triangle ABE \cong \triangle ABD$, and $\triangle AEC \cong \triangle BDC$.

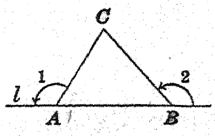
Proof:

STATEMENTS	REASONS
1. Let AE and BD bisect $\angle$s A and B, respectively.	1. *An angle has one and only one bisector.*
2. In $\triangle$s ABE and ABD, $\angle BAD = \angle EBA$.	2. *Given.*
3. $\angle 1 = \angle 2$.	3. *If equals are divided by equals the quotients are equal.*
4. $AB = AB$.	4. *The same line.*
5. $\triangle ABE \cong \triangle ABD$.	5. *a.s.a. = a.s.a.*
6. $\angle 5 = \angle 6, AE = BD$.	6. *Corresponding parts of congruent figures are equal.*
7. $\angle 7 = \angle 8$.	7. *Equal angles have equal supplements.*

STATEMENTS	REASONS
8. $\angle 3 = \angle 4$.	8. *If equals are divided by equals the quotients are equal.*
9. $\triangle AEC \cong \triangle BDC$.	9. *a.s.a.* = *a.s.a.*
10. $AC = BC$.	10. *Corresponding parts of congruent figures are equal.*

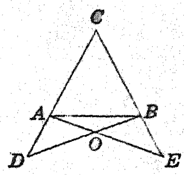
77. COROLLARY. *If a triangle is equiangular, it is also equilateral.*

EXERCISES



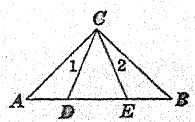
1. At points A and B on line l, $\angle s$ 1 and 2 are drawn each equal to 130°. What kind of triangle is ABC?

2. In $\triangle ABC$ when CA is extended through A and CB is extended through B, equal angles are formed with AB. Prove that $\triangle ABC$ is isosceles.



3. In $\triangle ABC$, $\angle BAC = \angle CBA$. Side CA is produced to D and side CB to E so that $AD = BE$. Why is $CD = CE$?

4. In Ex. 3 prove that $DB = EA$.



5. Given $\angle A = \angle B$, $\angle 1 = \angle 2$. Prove that $DC = EC$.

6. Give two converses for the statements in Ex. 5. Prove one of them.

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7. If the base of an isosceles triangle is trisected, and each point of trisection is joined to the vertex, prove that the triangle so formed is isosceles.

8. From any point in the bisector of an angle a line is drawn making equal angles with the sides of the angle. Prove that the triangle thus formed is isosceles.

9. In Ex. 3, prove that $\triangle AOB$ is isosceles.

***10.** In the figure for Ex. 3, $\angle BAC = \angle CBA$, AE bisects $\angle DAB$ and DB bisects $\angle ABE$; AE and BD meet at O. Prove that $\triangle DOE$ is isosceles.

78. More tools for proving segments and angles equal. Sections 69 and 76 give us two new ways to prove segments and angles equal.

Tool IV. *To prove two segments equal, prove them sides opposite equal angles in a triangle.*

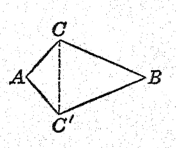
Tool V. *To prove two angles equal, prove them base angles of an isosceles triangle.*

TWO MORE CONGRUENCE THEOREMS

79. Two triangles having their sides respectively equal. Construct two triangles with sides 3 in., $2\frac{1}{2}$ in., and 2 in. Call one triangle ABC and the other one $A'B'C'$. Superpose $\triangle A'B'C'$ on $\triangle ABC$. Do you think that two triangles are congruent if their sides are respectively equal?

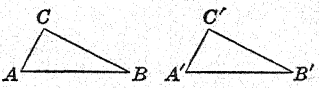
If you imagine $\triangle A'B'C'$ placed on $\triangle ABC$, can you make $A'B'$ coincide with AB? Why? What angles must be equal to make $A'C'$ take the direction of AC? Are you told that the angles are equal?

Since superposition will not do, work the following exercises and see if you can discover a method of proving the triangles congruent.



Ex. 1. In the kite shown in the figure, $AC = AC'$ and $CB = C'B$. Prove that $\angle ACB = \angle BC'A$.

Ex. 2. If $AC = A'C'$, $BC = B'C'$, $AB = A'B'$ how can you place $\triangle$s ABC and $A'B'C'$ so as to get the figure in Ex. 1?

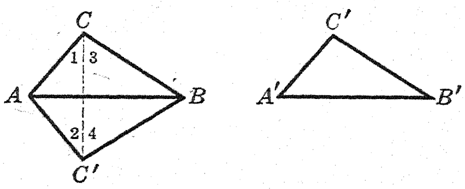


Ex. 3. What auxiliary line must be drawn?

Ex. 4. After proving $\angle C = \angle C'$ as in Ex. 1, how can you prove $\triangle ABC \cong \triangle ABC'$? (First figure.)

Proposition 5. Theorem

80. *If the sides of one triangle are equal, respectively, to the sides of another, the triangles are congruent.* (*s.s.s.* = *s.s.s.*)



Given: $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, $BC = B'C'$.

To prove: $\triangle A'B'C' \cong \triangle ABC$.

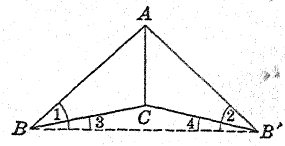
Plan: Place $\triangle A'B'C'$ next to $\triangle ABC$ so that $A'B'$ coincides with AB, A' falling on A and C and C' on opposite sides of AB. Show that $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and hence $\angle C = \angle C'$.

Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ next to $\triangle ABC$ so that $A'B'$ coincides with its equal AB, A' falling on A, and C and C' on opposite sides of AB. Draw CC'.	1. *Post.* 7. *Given* $AB = A'B'$.
2. In $\triangle ACC'$ $\angle 1 = \angle 2$.	2. *Given* $AC = A'C'$ (§ 69).
3. In $\triangle BCC'$ $\angle 3 = \angle 4$.	3. *Given* $BC = B'C'$. (§ 69).
4. $\angle 1 + \angle 3 = \angle 2 + \angle 4$.	4. *Ax.* 2.
5. $\angle C = \angle C'$.	5. *Ax.* 7.
6. $\triangle ABC \cong \triangle ABC'$.	6. *s.a.s.* = *s.a.s.*
7. $\triangle ABC \cong \triangle A'B'C'$	7. *Ax.* 7.

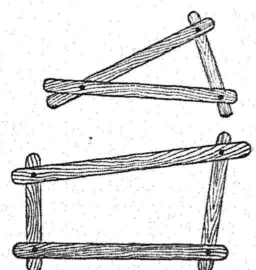
Note. — Reasons must be stated in full.

Discussion: What difference is there in the proof, if $A'C'$ is made to coincide with AC? Why is $\angle 1 = \angle 2$? Why is $\angle 3 = \angle 4$? Then why is $\angle ABC = \angle A'B'C'$? Write out the proof, using this figure.



EXERCISES

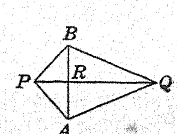
1. If you nail three strips of wood together so as to form a triangle, using only one nail at each joint, is the frame rigid, or can it be changed into different shapes by exerting pressure upon it?



In the same way, if you nail four strips of wood together, as shown in the drawing, can this frame so made be changed into different shapes by exerting pressure upon it?

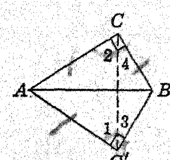
Show that the fact that a triangular frame is rigidly fixed when its three sides are of fixed length follows from Prop. 5.

2. In the kite shown, if $\triangle APB$ and AQB are isosceles, can you prove that $\triangle PBQ \cong \triangle PAQ$?



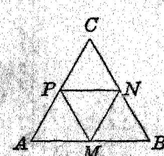
3. In Ex. 2, prove that PQ bisects $\angle APB$.

4. In the figure, $AC = AC'$, $\angle BCA = \angle AC'B = 90°$. Prove that $\angle 3 = \angle 4$ and hence that $BC = BC'$. Why then is $\triangle ABC \cong \triangle ABC'$?

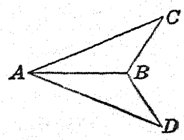


5. Given: Equilateral $\triangle ABC$. M, N, and P are mid-points of the sides.

To prove: $\triangle MNP$ is equilateral.



6. **Given:** $AC = AD$.
$BC = BD$.
To prove: AB bisects $\angle DAC$.



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7. Prove that the median to the base of an isosceles triangle bisects the vertex angle.

8. If the opposite sides of a quadrilateral (polygon having four sides) are equal, the opposite angles are equal.

*9. Prove that two right triangles are congruent if the hypotenuse (side opposite the right angle) and side of one are equal respectively to the hypotenuse and side of the other.

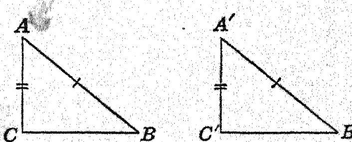
HINT. — Place the triangles as in the figure for Ex. 4.

*10. If the opposite sides of a quadrilateral (see Exercise 8) $ABCD$ are equal, the line segment connecting A and C is bisected by BD.

*11. Each of three students worked a problem at the blackboard. On taking their seats each discovered an error in a problem and each raised his hand to correct the error. Assuming that no one discovered an error in his own problem, what is the greatest possible number of problems correct?

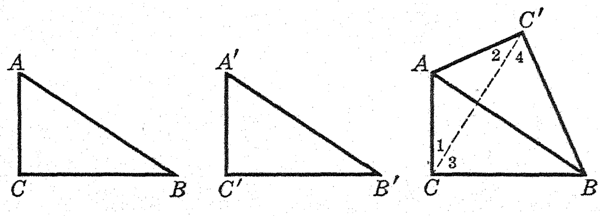
81. In a right triangle the side opposite the right angle is called the **hypotenuse;** the other sides are called the **legs.**

82. Congruent right triangles. ABC and $A'B'C'$ are right triangles with hypotenuse AB equal to hypotenuse $A'B'$; and leg AC equal to leg $A'C'$. If you place the triangles as in the figure for Ex. 4, page 82, do you think you can prove them congruent? Try before reading the proof on the next page.



Proposition 6. Theorem

83. *Two right triangles are congruent if the hypotenuse and a side of one are equal, respectively, to the hypotenuse and a side of the other.* (*hyp. side = hyp. side.*)



Given: Right $\triangle$ ABC and $A'B'C'$ with $\angle$ C and C' right angles, $AC = A'C'$, $AB = A'B'$.

To prove: $\triangle A'B'C' \cong \triangle ABC$.

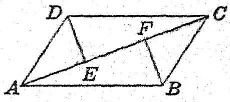
Plan:

1. Place the triangles as in § 80 and draw CC'.
2. Since $\triangle ACC'$ is isosceles, $\angle 1 = \angle 2$ (§ 69). Then $\triangle BCC'$ is isosceles (§§ 39 and 76).

Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ next to $\triangle ABC$ so that $A'B'$ coincides with its equal AB, A' on A. Let C and C' fall on opposite sides of AB. Draw CC'.	1. *Post.* 7. *Given* $AB = A'B'$.
2. $AC' = AC$.	2. *Given* $AC = A'C'$.
3. $\angle 1 = \angle 2$.	3. § 69.
4. $\angle 3 = \angle 4$.	4. § 39.
5. $BC = BC'$.	5. § 76.
6. $\therefore \triangle ABC \cong \triangle ABC'$.	6. *s.s.s. = s.s.s.*
7. $\therefore \triangle ABC \cong \triangle A'B'C'$.	7. *Ax.* 7.

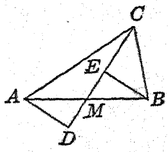
EXERCISES



1. $AD = BC$ and $\perp$s DE and BF to AC are equal. Prove that $AF = CE$.

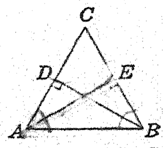
2. With the data given in Ex. 1, prove $DC = AB$.

3. Two triangles ABC and $A'B'C'$ have $AB = A'B'$, $\angle A = \angle A'$, and altitude $AD =$ altitude $A'D'$. Prove that $\triangle ABC \cong \triangle A'B'C'$.



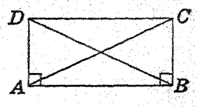
4. In $\triangle ABC$, CM is a median, $BE \perp CM$ at E, $AD \perp CM$ produced at D, and $BE = AD$. Prove that $EM = MD$.

5. If the altitudes to two sides of a triangle are equal, the triangle is isosceles.



6. The three altitudes of an equilateral triangle are equal.

7. Given $AE \perp BC$, and $BD \perp AC$; $AD = BE$. Prove that $\triangle ABC$ is isosceles.



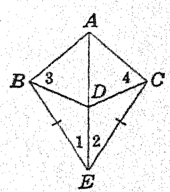
8. If $AC = BD$, and $\angle$s A and B are right angles, prove that $AD = BC$.

---

9. If the perpendiculars drawn from the points of trisection of a side of a triangle to the other two sides are equal, the triangle is isosceles.

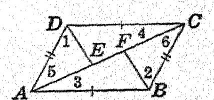
10. If from a point within an angle perpendiculars drawn to the sides of the angle are equal, the point lies on the bisector of the angle.

11. A line through any point within an angle, perpendicular to the bisector of the angle, makes equal angles with the sides.



***12. Given:** $BE = CE$.
ADE is a straight line.
$\angle 1 = \angle 2$.

To prove: $\angle 3 = \angle 4$.



***13.** If $AD = BC$, $AB = CD$, $\angle 1 = \angle 2$, and AC is a straight line, prove that $DE = BF$.

***14.** In the same figure if $AB = CD$, $AE = CF$, $\angle 3 = \angle 4$, then $\angle 1 = \angle 2$.

84. The proof of an original exercise can be discovered by analysis. You have seen that, in order to prove that two segments or two angles are equal, we try to show that they are corresponding parts of congruent triangles, or that they are sides or angles in an isosceles triangle. It sometimes happens that they cannot immediately be proved equal. A process for discovering with what step to begin a proof is called an **analysis.**

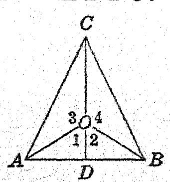
The steps of an analysis may be expressed symbolically as follows: If A is to be proved, reason thus: A will be true if B can be proved true; B will be true if C can be proved true; C follows if D can be proved true; but D is given true; hence begin by proving that C is true.

Read carefully the analysis of the following exercise.

Given: $AC = BC$, O is any point such that $AO = BO$.

To prove: $COD \perp AB$.

Analysis: 1. CD will be $\perp AB$ if $\angle CDA = \angle BDC$.



2. $\angle CDA$ will be equal to $\angle BDC$ if $\triangle ADO \cong \triangle BDO$.

3. $\triangle ADO$ will be congruent to $\triangle BDO$ if $\angle 1 = \angle 2$.

4. $\angle 1$ will be equal to $\angle 2$ if $\angle 3 = \angle 4$ (§ 40).

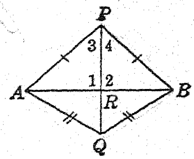
5. $\angle 3$ will be equal to $\angle 4$ if $\triangle AOC \cong \triangle BOC$.

6. But $\triangle AOC \cong \triangle BOC$. (s.s.s. = s.s.s.)

Now reverse the steps and, beginning with 6, prove $\triangle AOC \cong \triangle BOC$, thus making $\angle 3 = \angle 4$. Then show that $\angle 1 = \angle 2$ and hence $\triangle ADO \cong \triangle BDO$.

85. Tool VI. *If sufficient data are not given to prove two triangles directly congruent, prove the congruence of another pair which will give the needed data.*

EXERCISES



1. Given: $AP = PB, AQ = QB$.

To prove: $AR = RB$.

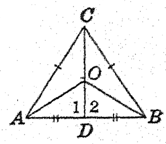
SUGGESTIONS. — Think: "1. I can prove $AR = RB$ if I can prove $\triangle ARP \cong \triangle BRP$.

2. I can prove $\triangle ARP \cong \triangle BRP$ if I can prove $\angle 3 = \angle 4$.

3. I can prove $\angle 3 = \angle 4$ if I can prove $\triangle APQ \cong \triangle BPQ$."

Reverse the steps and write the proof by first proving $\triangle APQ \cong \triangle BPQ$.

Write out an analysis of the following:

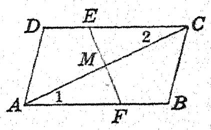


2. Given: $AC = BC$, $AD = DB$, O is any point on CD.

To prove: $AO = BO$.

SUGGESTIONS. — 1. Will AO equal BO if $\triangle ADO \cong \triangle BDO$?

2. Will $\triangle ADO$ be congruent to $\triangle BDO$ if $\angle 1 = \angle 2$? How can you prove $\angle 1 = \angle 2$?



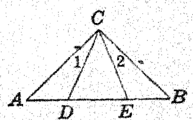
3. Given: $AD = BC$, $AB = DC$, $AM = MC$, EF and AC straight lines.

To prove: $EM = MF$.

SUGGESTIONS. — Think: "1. EM will equal MF if I can prove $\triangle AMF \cong \triangle CME$.

2. These ⚠ will be congruent if I can prove $\angle 1 = \angle 2$.

3. $\angle 1$ will equal $\angle 2$ if what triangles are congruent?"



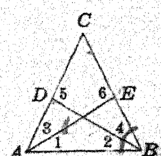
4. Given: $AC = BC$, $AE = DB$.

To prove: $\triangle DCE$ isosceles.

5. In the same figure:

Given: $AC = BC$, $\angle 1 = \angle 2$.

To prove: $AE = BD$.



6. Given: $\angle 1 = \angle 2$, $AE = DB$.

To prove: $\angle 3 = \angle 4$.

7. In the same figure:

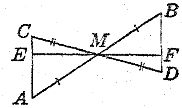
Given: $AC = BC$, $\angle 1 = \angle 2$.

To prove: $AD = BE$.

8. In the same figure:

Given: $AC = BC$, AE bisects $\angle A$ and BD bisects $\angle B$.

To prove: $AD = BE$.

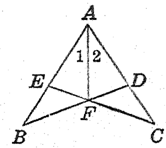


9. Given: $AM = MB$, $CM = MD$; AB, CD, and EF are straight lines.

To prove: $EM = MF$.

***10. Given:** $AE = AD$, $AB = AC$; BD and CE are straight lines.

To prove: $\angle 1 = \angle 2$.



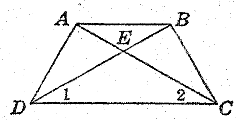
***11.** In the same figure:

Given: $AB = AC$, $BF = FC$, and BD and CE are straight lines.

To prove: $AE = AD$.

***12. Given:** $AD = BC$, $BD = AC$.

To prove: $\triangle DEC$ is isosceles.



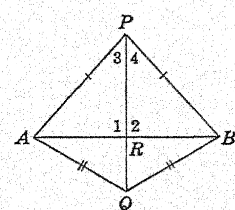
***13.** In Ex. 12 interchange "$AD = BC$" with "$\triangle DEC$ is isosceles" and prove the converse thus formed.

Theorems About Perpendiculars and Perpendicular Bisectors

86. The next theorem is: *If two points are each equidistant from the ends of a segment, they determine the perpendicular bisector of the segment.*

Given: Segment AB; points P and Q with $PA = PB$ and $QA = QB$, PQ intersects AB at R.

To prove: $AR = RB$, and $PR \perp AB$.



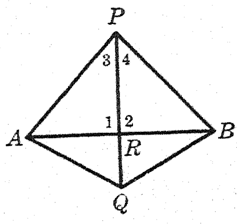
Analysis: Think: "1. AR will be equal to RB, and $\angle 1 = \angle 2$ if I can prove $\triangle ARP \cong \triangle BRP$.

2. I can prove $\triangle ARP \cong \triangle BRP$, if I can prove $\angle 3 = \angle 4$.

3. I can prove $\angle 3 = \angle 4$ if I can prove $\triangle APQ \cong \triangle BPQ$."

PROPOSITION 7. THEOREM

87. *If two points are each equidistant from the ends of a segment, they determine the perpendicular bisector of the segment.*



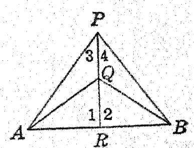
Given: Segment AB; P and Q with $PA = PB$, $QA = QB$; PQ intersects AB in R.

To prove: $AR = RB$, $PQ \perp AB$.

Plan:

1. Prove $\triangle AQP \cong \triangle BQP$. — s.s.s. = s.s.s.
2. Why does $\angle 3 = \angle 4$? — § 58.
3. Prove $\triangle ARP \cong \triangle BRP$. — s.a.s. = s.a.s.
4. If $\angle 1 = \angle 2$, is $PR \perp AB$? Why?

Write the proof in full.



Discussion: Could the figure at the right be used in the proof? Why is $\triangle AQP \cong \triangle BQP$? Then what angles are equal? Give the proof.

88. Tool VII. *Prove two angles are right angles by showing that they are equal supplementary adjacent angles.*

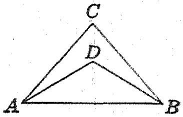
Tool VIII. *Prove that a line is the perpendicular bisector of a segment by proving that two of its points are equidistant from the ends of the segment.*

Nearly every new proposition will give you an additional tool that you can use in solving original exercises. Make a list of them as you study geometry.

EXERCISES

1. Given: In the figure $AC = BC$, and $AD = BD$.

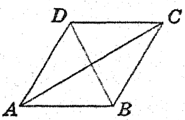
To prove: CD produced is the perpendicular bisector of AB.



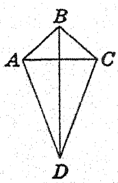
2. Complete the theorem proved in Ex. 1: *If two isosceles triangles are constructed on the same base, the line joining their vertices . . .*

3. Given: $AB = BC = CD = DA$.

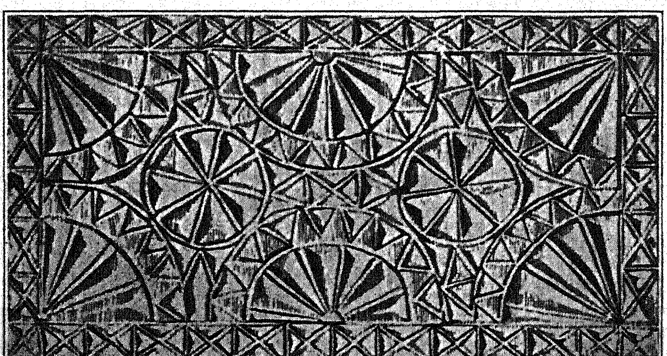
To prove: $AC \perp$ bis. of BD.
$BD \perp$ bis. of AC.



4. Complete the theorem for Ex. 3: *If the sides of a quadrilateral are equal, each diagonal is the . . .*



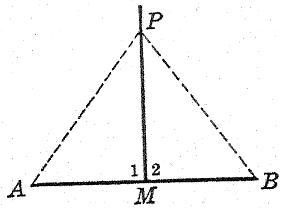
5. Write a theorem about the diagonals of the kite at the right, then prove it.



GEOMETRIC DESIGNS IN WOOD CARVING

PROPOSITION 8. THEOREM

89. *a. Any point on the perpendicular bisector of a segment is equidistant from the ends of the segment.*



Given: P, any point on PM the perpendicular bisector of AB.

To prove: $PA = PB$.

Plan: Prove $\triangle APM \cong \triangle BPM$.

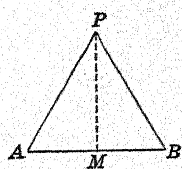
Proof: Write the proof.

b. (Conversely). *Any point equidistant from the ends of a segment is on the perpendicular bisector of the segment.*

Given: AB and point P, $PA = PB$.

To prove: P is on the $\perp$ bis. of AB.

Plan:



Think: "I can prove P is on the perpendicular bisector of AB by drawing PM to the mid-point of AB and using the theorem, *If two points are each equidistant from the ends of a segment . . .*"

Proof: Write in full.

EXERCISES

1. Draw a segment AB. Mark a point that is the same distance from A and B. In a similar way mark four other points, some above and some below AB. On what line do they lie?

2. A robber hid some treasure by the side of a road and equidistant from two trees. Show how to find it.

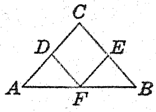
3. How many isosceles triangles can you draw having the same base? Where will the vertices lie? Draw a sketch showing positions of the vertices, four above and four below AB.

4. Find a point on side AC (or AC extended) of $\triangle ABC$, equidistant from A and B. Find such a point also on side BC.

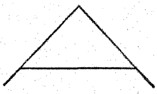
5. The perpendicular bisector of side AB of $\triangle ABC$ intersects AC in D, and BC produced in E. Prove that $EB = EA$ and that $BD = DA$.

MISCELLANEOUS EXERCISES

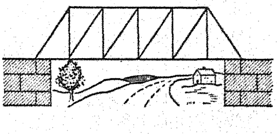
1. Are all the bisectors of the angles of an equilateral triangle equal? Prove it.



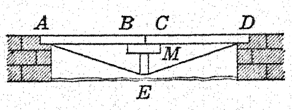
2. If $AC = BC$, and D, E, and F are the mid-points of the sides, prove $FD = FE$.



3. Why is a roof sufficiently braced by a board nailed across each pair of rafters?



4. Why is a span of a bridge in which the supporting frame is made as shown in the drawing, sufficiently supported?



5. The drawing shows a part of a bridge, called an inverted king-post truss. AB and CD are steel bars resting on a plate M. The plate M is supported by wire cables EA and ED. Why will this structure support a heavy weight?

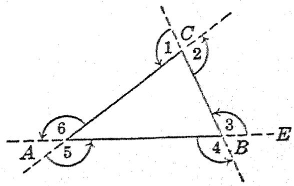
6. $\triangle\triangle$ ABC and $A'B'C'$ have $AC = A'C'$, $AB = A'B'$, and median CM = median $C'M'$. Prove that $\triangle ABC \cong \triangle A'B'C'$.

HINT. — First prove $\triangle\triangle$ AMC and $A'M'C'$ congruent.

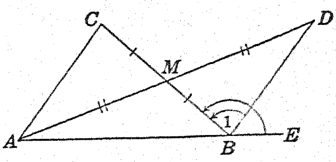
7. How would you construct a line through a given point within a given acute angle, so as to form an isosceles triangle with the sides of the angle?

HINT. — First bisect the given angle. What relation would the required line have to the bisector?

90. An **exterior** angle of a polygon is an angle formed by one side produced and the adjacent side. A triangle thus has six exterior angles which are equal in pairs.



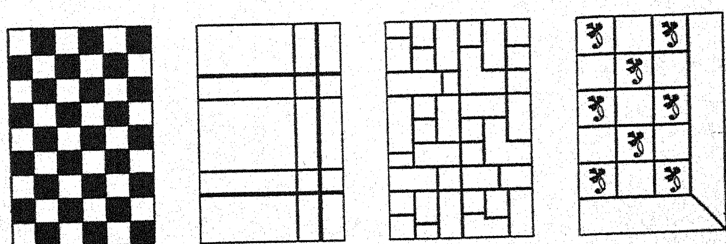
$\measuredangle$ BAC and ACB are spoken of as the interior angles opposite the exterior angles at B. Do you think $\angle EBC$ is greater than ($>$) either $\angle BAC$ or $\angle ACB$?



In $\triangle ABC$, if M is the midpoint of BC and $AM = MD$, why is $\triangle AMC \cong \triangle BDM$? Why is $\angle 1 = \angle C$?

If AB is produced to E, an exterior angle EBC is formed. Why is $\angle EBC > \angle 1$ (Ax. 6)?

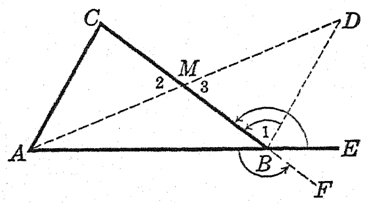
Then is $\angle EBC > \angle C$ (Ax. 7)?



GEOMETRIC DESIGNS IN LINOLEUM

PROPOSITION 9. THEOREM

91. B. *An exterior angle of a triangle is greater than either opposite interior angle.*



Given: $\triangle ABC$, with exterior angle EBC.

To prove: $\angle EBC > \angle A$ and $\angle EBC > \angle C$.

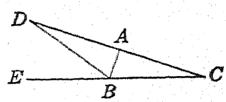
Plan: Prove $\triangle ACM \cong \triangle BDM$ and then compare $\angle EBC$ with $\angle 1$ and $\angle C$.

Proof:

STATEMENTS	REASONS
1. Let AMD bisect CB; make $AM = MD$. Draw BD.	1. *Post.* 8.
2. $AM = MD, CM = MB$.	2. *Construction.*
3. $\angle 2 = \angle 3$.	3. § 46.
4. $\triangle ACM \cong \triangle BDM$.	4. *s.a.s.* = *s.a.s.*
5. $\angle 1 = \angle C$.	5. § 58.
6. But $\angle EBC > \angle 1$.	6. *Ax.* 6.
7. $\therefore \angle EBC > \angle C$.	7. *Ax.* 7.

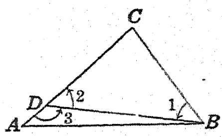
In the same way, by drawing a line from C through the midpoint of AB, prove that $\angle ABF > \angle A$. Then, since $\angle ABF = \angle EBC$, $\angle EBC > \angle A$. Write the proof in full.

EXERCISES



1. In the figure name an exterior angle of $\triangle ABC$; of $\triangle DBC$; of $\triangle ADB$.

2. Compare $\angle DAB$ and $\angle ACB$; $\angle DBE$ and $\angle BDA$.



3. Name an exterior angle of $\triangle ABD$; of $\triangle DBC$.

4. Compare $\angle 2$ and $\angle A$; $\angle 3$ and $\angle C$.

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5. If $DC = BC$, show that $\angle CBA > \angle A$

CONSTRUCTIONS

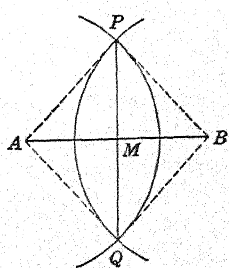
In Unit I you made several geometric constructions. We shall now prove that these constructions are correct.

CONSTRUCTION I

92. *To construct the perpendicular bisector of a segment.*

Given: Segment AB.

Required: Construct $\perp$ bis. of AB.



Construction: 1. With A as a center and with any radius more than half AB, draw an arc.

2. With B as a center and with the same radius, draw an arc intersecting this arc in P and Q.

3. Draw PQ intersecting AB in M.

4. PQ is the required line.

Proof:

STATEMENTS	REASONS
1. $PA = PB, QA = QB$.	1. *Equal circles have equal radii.*
2. PQ is the $\perp$ bis. of AB.	2. *If two points are equidistant from the ends of a segment they determine the perpendicular bisector of the segment.*

Of course this construction includes the construction studied in § 21: *To bisect a line segment.*

In a similar way prove the correctness of the constructions below.

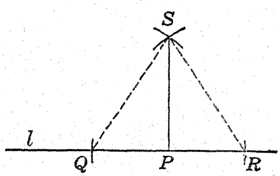
Construction II

93. *To construct an angle equal to a given angle.*

In § 24 a construction is given for copying an angle equal to a given angle. To prove that $\angle A' = \angle A$, draw PQ and $P'Q'$ and prove that $\triangle PAQ \cong \triangle P'A'Q'$, and hence that $\angle A' = \angle A$. (See figure in § 24.)

Construction III

94. *To construct a perpendicular to a line at a given point on the line.*



Given: Line l; point P on l.

Required: Construct a $\perp$ to l at P.

Construction: 1. With P as center and any radius draw an arc intersecting l at Q and R.

2. With Q and R as centers and any radius more than half QR draw arcs intersecting at S.

3. Draw SP.

4. SP is the required perpendicular.

Write the proof.

Hint. — Use § 87.

Construction IV

95. *To construct a perpendicular to a given line from a given outside point.*

Given: Line AB and outside point P.

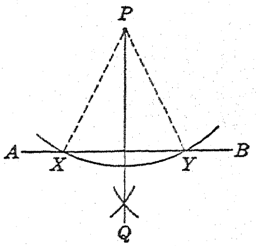
Required: To construct a perpendicular from P to AB.

Construction: 1. With P as center and any radius sufficiently great, draw an arc cutting AB at X and Y.

2. With X and Y as centers and any radius more than half XY draw arcs intersecting at Q.

3. Draw PQ.

4. Then PQ is the required perpendicular.



Proof: Use § 87.
Write the proof in full.

Construction V

96. *To construct the bisector of an angle*

Prove that the construction given for bisecting an angle (§ 33) is correct.

Suggestion. — Draw QS and RS and then prove $\triangle PQS \cong \triangle PRS$.

The following theorem is based on Proposition 9 and Constructions III and IV:

97. B. *One and only one perpendicular can be drawn to a given line through a given point.* (Prop. 10.)

Proof: Let PQ be $\perp l$ and PR be any other line through P. Then in Fig. 1, $\angle 1 > \angle 2$ (§ 91). But $\angle 1 = 90°$ ($PQ \perp l$). Therefore PR is not perpendicular to l. (§ 26.)

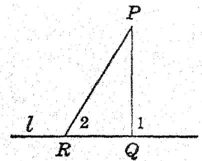


Fig. 1

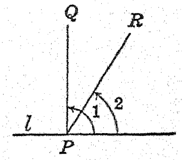
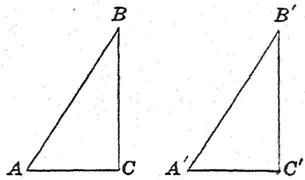


Fig. 2

In Fig. 2, $\angle 1 > \angle 2$ (Ax. 6). But $\angle 1 = 90°$. (§ 26.) Hence PR is not perpendicular to l. (§ 26.)

There follows from these a fifth congruence theorem. The proof is by superposition. Read it carefully.

98. B. *Two right triangles are congruent if the hypotenuse and adjacent angle of one are equal, respectively, to the hypotenuse and adjacent angle of the other.* (Prop. 11.)



Given: $\triangle$s ABC and $A'B'C'$; $\angle$s C and C' right angles; $AB = A'B'$; $\angle A = \angle A'$.

To prove: $\triangle A'B'C' \cong \triangle ABC$.

Plan: Superposition.

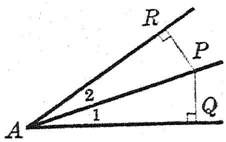
STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with its equal $\angle A$, $A'C'$ falling along AC and $A'B'$ along AB.	1. *Post.* 7.
2. B' will fall on B.	2. $A'B' = AB$, *given.*
3. $B'C'$ will fall along BC.	3. § 97.
4. C' will fall on C.	4. *Post.* 2.
5. $\triangle A'B'C' \cong \triangle ABC$.	5. § 54.

99. By the **distance from a point to a line** we mean the distance measured on the perpendicular from the point to the line.

> It is not possible to find in all geometry more difficult and more intricate questions or more simple and lucid explanations than those given by Archimedes. Some ascribe this to his natural genius; while others think that incredible effort and toil produced these, to all appearance, easy and unlaboured results. No amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required. — Plutarch
>
> Life of Marcellus (Dryden)

PROPOSITION 12. THEOREM

100. *a. Any point in the bisector of an angle is equidistant from the sides of the angle.*



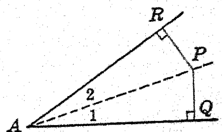
Given: $\angle A$, P any point on AP the bisector of $\angle A$; $PQ \perp AQ$, $PR \perp AR$.

To prove: $PQ = PR$.

Plan: Prove $\triangle APR \cong \triangle APQ$.

Proof: Write in full.

b. (Conversely.) *Any point equidistant from the sides of an angle is on the bisector of the angle.*



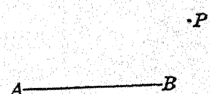
Given: $\angle A$, and point P with $PR \perp AR$ and $PQ \perp AQ$, $PR = PQ$.

To prove: P lies on the bisector of $\angle A$.

Plan: Draw AP and prove the triangles congruent.

EXERCISES

1. Draw a large acute triangle and construct its three altitudes.



2. Draw a line AB on your paper and from a point P placed as at the right construct a perpendicular to AB.

3. Draw a large obtuse triangle and construct its three altitudes. Do the altitudes from the vertices of the acute angles meet the *opposite side*, or the *opposite side produced*? Do the altitudes seem to be concurrent?

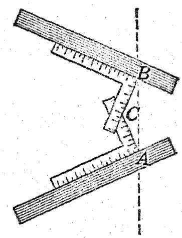
4. Draw a right triangle. Construct the altitude from the vertex of one of the acute angles. Where does it fall?

5. Draw an angle and construct an angle equal to it.

6. Draw an angle and construct lines dividing it into four equal parts.

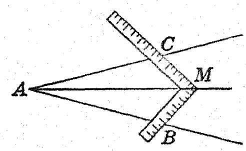
7. Draw a large triangle and construct the perpendicular bisectors of its sides.

8. Construct the medians of a triangle. Do they seem to be concurrent?

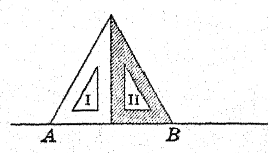


9. If a carpenter wishes to mark a line that will cut two converging boards at equal angles, he places two steel squares against the boards, as shown in the drawing, and adjusts them so that $AC = BC$. Prove that the line AB then makes equal angles with the edges of the two boards.

10. A carpenter can bisect an angle BAC with his steel square by marking off AC and AB of equal length on the sides of the angle, and then placing his square so that $MC = MB$, as shown in the drawing. Then he marks the point M at the heel of the square, and draws AM. Prove that AM bisects $\angle BAC$.



11. The accuracy of the right angle of a triangle can be tested by drawing a perpendicular to a line AB, using position I as in the figure, then turning the triangle over to position II to see if the perpendicular coincides with the edge of the triangle in its new position. Why should they?



Indirect Proof

101. In the *direct proof*, which we have used thus far, we have put together known truths, (the hypothesis, definitions, axioms, previously proved theorems, etc.) and have thus, step by step, developed a proof.

The **indirect method** of proof is a kind of proof that we frequently use in everyday reasoning. See how it is used in the following example:

Charles and Jack were playing ping-pong. Charles claimed that a ball that he served hit the further edge of the table top. The boys agreed that the ball passed over the center of the net and did not touch it. Charles used the indirect method of proof in showing that his contention was correct:

Charles: The ball either hit the table top, or else it did not.

Jack: Those are the only possibilities.

Charles: You claim that it did not. Let us suppose that you are right and that the ball did not touch the table on your side. Then it would have gone straight until it hit the wall or floor.

Jack: I must admit that.

Charles: But it did not go straight. It was sharply deflected to the left. Hence it must have been deflected by something. You agree that there was nothing but the table top to deflect it.

Jack: You win! It's your point.

102. Charles used three important features of the method of indirect proof.

When he said: " The ball either hit the table top or it did not " he used the principle:

1. *Of two contradictory propositions, one must be true and the other must be false.* There is no middle ground, no third possibility. In logic this is called the *law of excluded middle.*

2. When he said: " Let us suppose that you are right " he was assuming, temporarily, the truth of a statement he felt to be false.

Finally he showed that Jack was wrong by the principle:

3. *If the conclusion of a correct line of reasoning is shown to be false, then the hypothesis from which the conclusion follows must be false.*

Thus, from the hypothesis that the ball did not touch the table we have the conclusion: *then it must have gone straight.* But this conclusion is known to be false. Hence the hypothesis must be false.

> **Euclid** (330–275 B.C.) was the first author of a textbook on geometry, which was a part of a book called *The Elements.* He used the *indirect proof* in proving his sixth proposition, which was, "If two angles of a triangle are equal, the sides opposite are equal."

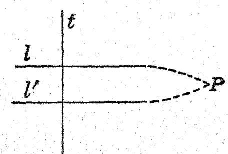
103. Indirect proof in geometry. *Two lines perpendicular to the same line do not intersect.*

Given: $l \perp t, l' \perp t$.

To prove: l does not intersect l'.

Plan: Indirect method.

Proof:

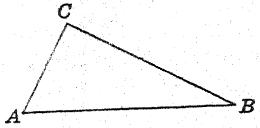


STATEMENTS	REASONS
1. Either l does not intersect l' or l intersects l'.	1. § 102 (1).
2. Assume that l intersects l' at P. Then from P there are two perpendiculars to t. But this is impossible.	2. § 97.
3. Hence the hypothesis "l intersects l'" is false.	3. § 102 (3).
4. Therefore l does not intersect l'.	4. § 102 (1).

EXERCISES

Prove the following by the indirect method.

1. If two angles of a triangle are unequal the sides opposite are unequal.



Given: $\triangle ABC$; $\angle A \neq \angle B$.

To prove: $AC \neq BC$.

HINT. — Either $AC = BC$, or $AC \neq BC$. Show that the assumption "$AC = BC$" leads to a contradiction of the given fact, $\angle A \neq \angle B$.

2. If two sides of a triangle are unequal, the angles opposite are unequal.

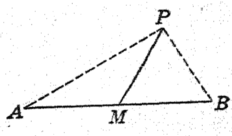
3. A triangle cannot have two right angles.

HINT. — It either has two right angles or it has not. Show why the first possibility contradicts § 97.

4. A base angle of an isosceles triangle cannot be a right angle.

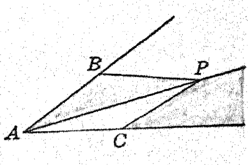
5. If one angle of a triangle is 100°, the other angles must each be less than 80°. (Draw the exterior angle at the vertex of the 100° angle and use § 91.)

6. If M is the mid-point of AB and PM is not $\perp$ AB, prove PA is not equal to PB. (Draw PA and PB and suppose they are equal. Then show by congruent triangles that your hypothesis is contradicted.)



7. **Given:** $AB = AC$.
BP is not equal to PC.

To prove: AP does not bisect $\angle A$.



8. If a triangle has one obtuse angle, the other angles are acute. (Assume that it has two obtuse angles and use § 91.)

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9. If a triangle is not isosceles, the bisector of an angle is not perpendicular to the opposite side.

10. If the median to the base of a triangle does not meet the base at right angles, the other two sides are not equal.

11. If two sides of one triangle are equal, respectively, to two sides of another triangle, prove that, if their included angles are not equal, then their third sides are not equal.

***12.** Prove the theorem in § 76 by the indirect method.

Suggestions. — Either $CA = CB$ or $CA \neq CB$. If $CA \neq CB$, either $CA < CB$ or $CA > CB$. If $CA < CB$, take CX on $CB = CA$. Then what do you know about $\angle\!\!s$ XAC and CXA (§ 69)? About $\angle\!\!s$ CXA and CBA (§ 91)? Then what contradiction is involved in the comparison of $\angle XAC$ and $\angle BAC$ (Ax. 6)?

***13.** See if you can figure this out: Three students, A, B, and C sat talking. Two were sophomores and one was a freshman. The following conversation took place:

A. (Makes a remark that B does not hear.)
B. (Speaking to C) "What did A say?"
C. "A said that he was a freshman."

Assuming that the sophomores always tell the truth and that the freshman never does, which of the three was a freshman?

104. Parallel (||) **lines** are lines that lie in the same plane and do not meet however far produced.

There are lines which do not meet and still are not parallel. Thus, the line between the front wall of a room and the floor, and that between the side wall and ceiling do not meet and are not parallel. But, of course, they are not *in the same plane.* Can you hold two rulers so that they will not meet (even if produced), and are not parallel?

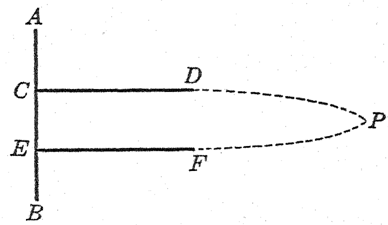
105. Postulate 11. *Two lines in the same plane must either be parallel or they must intersect.*

106. Postulate 12. *Through a given outside point there can be one and only one parallel to a given line.*

Note. — The words, *in the same plane,* are always understood in plane geometry.

PROPOSITION 13. THEOREM

107. *Two lines in the same plane perpendicular to the same line are parallel.*



Given: $CD \perp AB$; $EF \perp AB$.

To prove: $CD \parallel EF$.

Plan: Use the indirect method.

Proof:

STATEMENTS	REASONS
1. CD and EF either meet or are $\parallel$.	1. § 105.
2. Suppose they meet at point P. Then from P there are two perpendiculars to AB.	2. *Given*, $CD \perp AB$, $EF \perp AB$.
3. But this is impossible.	3. § 97.
4. $\therefore CD \parallel EF$.	4. § 105.

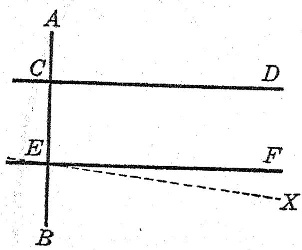
108. The next theorem is the converse of the one above.

PROPOSITION 13	CONVERSE
Given: $CD \perp AB$ $EF \perp AB$	**Given:** $CD \perp AB$ $CD \parallel EF$
To prove: $CD \parallel EF$.	**To prove:** $EF \perp AB$

In proving this converse use the indirect method.

Proposition 14. Theorem

109. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Given: $CD \parallel EF$; $CD \perp AB$.

To prove: $EF \perp AB$.

Plan: Indirect method of proof.

Proof:

STATEMENTS	REASONS
1. EF is either $\perp AB$ or it is not $\perp AB$.	1. *Only two possibilities.*
2. If EF is not $\perp AB$, suppose that $EX \perp AB$.	2. § 97.
3. Then $EX \parallel CD$.	3. § 107.
4. But $EF \parallel CD$.	4. *Given.*
5. It is impossible for both EF and EX to be $\parallel CD$.	5. § 106.
6. Hence our assumption that EF is not $\perp AB$ is wrong, and the only other possibility, $EF \perp AB$ must be true.	6. § 102 (3).

Discussion: You have now seen several examples of the use of the *indirect method of proof*. The method really consists of

supposing that the theorem is *not* true and then proving that the supposition leads to a contradiction of a known truth. For this reason this method of proof has been called by the Latin term, **reductio ad absurdum,** meaning reduction to an absurdity.

110. Summary of the work of Unit Two. In Unit Two you have learned:

I. ***You can prove two triangles are congruent by proving that:***

1. *They have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other.*
2. *They have two angles and the included side of one equal, respectively, to two angles and the included side of the other.*
3. *They have the sides of one equal, respectively, to the sides of the other.*
4. *They are right triangles having the hypotenuse and a side of one equal to the hypotenuse and a side of the other.*
5. *They are right triangles having the hypotenuse and an adjacent angle of one equal to the hypotenuse and an adjacent angle of the other.*

II. ***You can prove any two segments are equal by proving that:***

1. *They are corresponding sides of congruent figures.*
2. *They are sides opposite equal angles in the same triangle.*

III. ***You can prove any two angles are equal by proving that:***

1. *They are corresponding angles of congruent figures.*
2. *They are complements or supplements of equal angles.*
3. *They are base angles in an isosceles triangle.*
4. *They are vertical angles.*
5. *They are equal to a straight angle, or a right angle.*

IV. ***You can prove any two lines are parallel by proving that:***

1. *They are perpendicular to the same line.*

V. ***To prove that the methods used in Unit One for making the following constructions are correct:***

1. *To construct the perpendicular bisector of a segment.*
2. *To construct an angle equal to a given angle.*
3. *To construct a perpendicular to a line at a given point of the line.*
4. *To construct a perpendicular to a given line from a given outside point.*
5. *To construct the bisector of an angle.*

VI. ***The meaning of converse theorems.*** § 74.

VII. ***The use of geometric analysis.*** § 84.

VIII. ***Indirect proof and how to use it.*** §§ 101–102.

REVIEW OF THE SECOND UNIT

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the tests. The references given are those most closely related to the exercise.

1. Write the five theorems which may be used to prove triangles congruent. §§ 64, 65, 80, 83, 98.

2. Which of the theorems that you have had can be used to prove two angles equal? List them and then consult the following sections to see if you have them all: §§ 64, 65, 69, 80, 83, 98.

3. What theorem can be used to prove angles unequal? § 91.

4. See if you can list seven theorems to prove that two line segments are equal. §§ 64, 65, 76, 80, 83, 89, 98.

5. Of what does a demonstration consist? § 63.

6. See how many of the study aids given in § 66 you can write out. Read the section over again and correct your list.

7. What is meant by a geometric proof? § 63.

8. What is the side opposite the right angle in a right triangle called? What are the other sides called? § 81.

9. What are the legs of an isosceles triangle? § 14.

10. Draw a triangle and construct one of its medians. How many has it? § 73.

11. Draw an obtuse triangle and construct its three altitudes. § 95.

12. How many conditions can you impose on an auxiliary line that you construct? § 68.

13. State four pairs of conditions such that any one of the pairs may be imposed on an auxiliary line.

14. Write out the methods that may be used to prove segments and angles equal. Consult § 110, II, III, to see if you have them all.

15. Write out the theorems about perpendiculars. §§ 87, 97, 107, 109.

16. Tell how an analysis of an exercise is made. How do you obtain a proof from an analysis? § 84.

17. What are the three general steps in an indirect proof? § 102.

18. How does an indirect proof differ from a direct proof? § 101.

19. On what two principles does an indirect proof depend? § 102.

20. Write the two postulates about parallel lines. §§ 105, 106.

21. *First,* write each theorem indicated, in the form using "if" and "then." *Second,* indicate the hypothesis and conclusion. *Third,* write the converse or converses of the theorem.

a. The theorem in Ex. 11, page 112.
b. The theorem in Ex. 12, page 112.
c. The theorem in Ex. 13, page 112.
d. A statement whose converse is true.
e. A statement whose converse is false.

Complete the following:

22. An exterior angle of a triangle........ § 91.

23. One and only one perpendicular .... § 97.

24. An equiangular triangle is also .... § 77.

25. If two angles of a triangle are equal .... § 76.

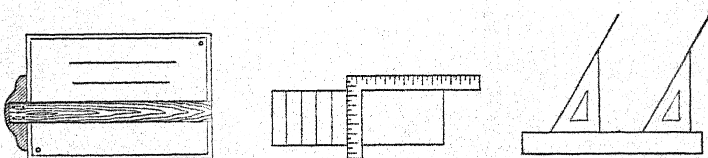
26. A corollary is a theorem .... § 70.

27. Can you draw one straight line through *any* three points? § 68.

28. From *any* given point P can you draw a line bisecting a given angle C? Why? § 68.

29. Can you construct from *any* given point P the perpendicular bisector of a given segment AB? Why? § 68.

For Ex. 30–32, see § 107.



30. What geometric principle is used in drawing parallel lines by means of a T-square?

31. What principle does a carpenter use when he marks off parallel lines on a board by moving one blade of his square along an edge and marking along the other?

32. Explain how parallel lines may be drawn with a drawing triangle. Why are the lines drawn parallel?

CONSTRUCTIONS

1. Construct an isosceles triangle whose vertex angle is 90°. What seems to be the sum of its angles?

2. Construct an angle of 45° by first constructing a perpendicular to a line and then bisecting the right angle.

3. Construct an angle of 135° by subtracting an angle of 45° from 180°. Use the method of § 24 to copy your angle.

4. Construct the sum of an obtuse angle and a straight angle.

5. Construct a triangle having a side of 2 in. and adjacent angles of 45°.

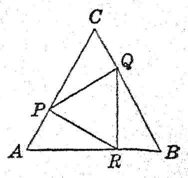
6. Draw a right triangle and construct its three altitudes. With what lines do two of them coincide?

7. Construct the perpendicular bisectors of the sides of an obtuse triangle.

8. Draw a triangle and construct the bisectors of its angles.

GENERAL EXERCISES

1. If the equal sides AC and BC of an isosceles triangle are produced through the vertex C to points D and E, respectively, so that $CD = CE$, prove that $DB = EA$.



2. If $\triangle ABC$ is equilateral, and $AP = BR = CQ$, prove that $\triangle PQR$ is equilateral. Write an analysis first.

3. In the same figure, if $\triangle PQR$ is equilateral, and $\angle PRA = \angle RQB = \angle QPC$, prove that $\triangle ABC$ is equilateral.

4. Two right triangles are congruent if the legs of one are, respectively, equal to the legs of the other.

5. Prove that if three altitudes of a triangle are equal, the triangle is equilateral.

6. Prove that the bisectors of the exterior angles at the base of an isosceles triangle form with the base an isosceles triangle.

7. The bisector of what angle of an isosceles triangle bisects the opposite side? Prove your answer.

8. Given two equilateral triangles on opposite sides of the same base. Prove that the line connecting their opposite vertices is the perpendicular bisector of the common base.

9. A median of a triangle is produced its own length through the side of the triangle and the extremity is connected to one of the other vertices of the triangle. Prove that a pair of the triangles formed are congruent.

10. If two medians of a triangle are equal and intersect so as to form with the third side an isosceles triangle, then the given triangle is isosceles.

11. If the diagonals of a quadrilateral bisect each other, the opposite sides are equal.

12. In two congruent triangles, prove that corresponding medians are equal.

13. A triangle is isosceles if perpendiculars drawn to the adjacent sides from any point in the altitude are equal.

14. The perpendicular bisectors of sides AB and BC of $\triangle ABC$ meet in O. Show that $OA = OB = OC$.

PRACTICE TESTS

These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT TWO

TEST ONE

Numerical Exercises

1. One angle of an equilateral triangle is 60°. How many degrees in each of the other angles? § 71.

2. ABC is an isosceles triangle with $AB = AC$. If $\angle B$ is 50°, how many degrees in the supplement of $\angle C$? § 69.

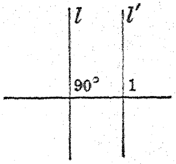
3. Three towns, A, B, and C, are so situated that C lies on the perpendicular bisector of the segment joining A and B. If C is 8 miles from A, how far is it from B? § 89.

4. An angle of a triangle is 36°. How large is the exterior angle at this vertex? § 91.

5. In $\triangle ABC$, $AB = 12$ in., $\angle B = 55°$, and $AC = 12$ in. How large is $\angle C$? § 69.

6. In $\triangle ABC$, $AB = 12$ in., $AC = 6$ in., $\angle A = 35°$, $BC = 8$ in. In $\triangle DEF$, $FE = 12$ in., $DE = 8$ in., and $DF = 6$ in. How large is $\angle F$? § 80.

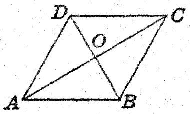
7. The angles of a triangle are 40°, 60°, and 80°. How many degrees in the largest exterior angle? § 91.



8. $l \parallel l'$. How large is $\angle 1$? § 109.

9. In $\triangle ABC$, $\angle A = 75°$ and $\angle C = 60°$. The number of degrees in the exterior angle at B must be greater than what number? § 91.

10. Triangle ABC has $AB = 14$ in., $\angle B = 45°$, $BC = 10$ in., and $\angle C = 90°$. Triangle DEF has $\angle D = 90°$, $\angle E = 45°$, and $EF = 14$ in. How long is DE? § 98.



11. $AB = BC = CD = AD$. If $DO = 6$ in., how long is DB? § 87.

12. C is on the perpendicular bisector of line segment AB. If $AC = 15$ in., how long is BC? § 89.

TEST TWO

Supplying Reasons

Supply axioms, postulates, definitions, or theorems as reasons for the statements below.

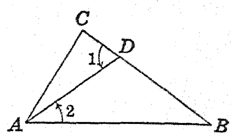
1. Triangles ABC and DEF are right triangles and AB and DE, the hypotenuses, are equal. If $AC = DF$, the triangles are congruent. § 83.

2. In $\triangle ABC$, $\angle s$ A and B each are complementary to 50°. Therefore triangle ABC is isosceles. § 76.

3. In $\triangle ABC$, with $BC = AC$, AM and BN are medians. Then $BM = AN$. Ax. 5.

4. If $\triangle s$ ABC and ABD have $AC = AD$ and $BC = BD$, they are congruent. § 80.

5. Given: $MA = MB$, $NA = NB$. **Conclusion:** $MN \perp AB$. § 87.



6. In the figure, $\angle 1 > \angle 2$. § 91.

7. In the same figure: **Given:** $\angle 2 = \angle B$. **Conclusion:** $AD = BD$. § 76.

8. Given: $\triangle ABC$; $AC = BC$; $AE \perp BC$, $BD \perp AC$. **Conclusion:** $\triangle ABD \cong \triangle ABE$. § 98.

9. Given: $PM \perp$ bisector of AB; Q is any point on PM. **Conclusion:** $QA = QB$. § 89.

10. Given: $l \parallel l'$; $l \perp t$. **Conclusion:** $l' \perp t$. § 109.

11. Given: $l \perp t, l' \perp t$. **Conclusion:** $l \parallel l'$. § 107.

12. Given: $PM \perp MR$. **Conclusion:** PR is not $\perp MR$. § 97.

TEST THREE

Multiple-Choice Statements

From the expressions printed in italics select that one which best completes the statement.

1. An exterior angle of a triangle is *an obtuse angle, supplementary to an opposite interior angle, greater than an opposite interior angle.* § 91.

2. The perpendicular to a segment at its mid-point *is an altitude, contains all points equidistant from the ends of the segment.* § 89.

3. In an obtuse triangle, *one, two, three* altitudes always lie outside the triangle.

4. In an isosceles triangle the angles opposite the equal sides are *supplementary, complementary, equal.* § 69.

5. The triangle is much used in constructing buildings, bridges, etc., because it is *easy to make, a rigid figure.* § 80.

6. The line from any vertex of a triangle to the mid-point of the opposite side is called the *perpendicular bisector, median, altitude.* § 73.

7. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is *isosceles, a right triangle.* § 65.

8. If the opposite sides of a quadrilateral are equal, a diagonal *bisects two angles of the quadrilateral, divides the quadrilateral into two congruent triangles, is perpendicular to the other diagonal.* § 80.

9. The number of converses a given theorem can have is *one, more than one, as many as you please.* § 74.

10. If a theorem is true, a converse of it *may, must* be true. § 74.

11. A point not on the perpendicular bisector of a segment *may be, is not* equidistant from the ends of the segment. § 89.

12. In an indirect proof a statement that contradicts the hypothesis is shown to be *false, true.* § 102.

CUMULATIVE TESTS ON THE FIRST TWO UNITS

TEST FOUR

Numerical Exercises

1. Triangle ABC has $AB = 16$ in., $BC = 7$ in., $CA = 14$ in., and $\angle C = 93°$. If $\triangle DEF$ has $DE = 7$ in., $\angle E = 93°$, and $EF = 14$ in., how large is DF? § 64.

2. If $\measuredangle x$ and y are complementary, and $\angle x - \angle y = 35°$, how large are x and y? § 38.

3. Four angles make up all the angular magnitude about a point. If the first is half the second, the third is 20° more than the second, and the fourth is 50° less than three times the third, how many degrees in each angle? § 34.

4. Through how many degrees does the minute hand of a clock revolve in one hour? § 34. 360°

5. How many straight lines can be drawn through two points? *Post.* 3.

6. How large is an angle which equals its complement? § 38.

7. How large is an angle which is 11 times its supplement? § 38.

8. If one of the four angles about a point made by two intersecting lines is 114° 48′, how large is each of the others? §§ 46, 38.

9. How many degrees in the angle formed by the hands of a clock at 6 o'clock? § 34. 180°

10. A certain angle contains a degrees. Give a formula for the number of degrees in its supplement. § 38.

11. What kind of an angle is less than its supplement? § 29.

12. Of two supplementary angles one is 6° less than twice as large as the other. How large is each? § 38.

TEST FIVE

Completing Statements

Complete the following statements.

1. A line drawn from a vertex of a triangle to the mid-point of the opposite side is a ———— of the triangle. § 73.

2. To construct a perpendicular to a segment at its extremity, I would first ———— the segment. § 94.

3. Two lines in the same plane can be proved parallel if they are ———— to the same line. § 107.

4. An ———— angle of a triangle is an angle formed by one ———— of a triangle and another side ————. § 90.

5. In studying a theorem in geometry I should attempt to write out the proof with the aid of the plan given, and without consulting the proof in the book. If I cannot do so, I should read over the ———— and try to supply the ———— without looking them up. § 66.

6. If two straight lines intersect, the ———— angles are equal. § 46.

7. An ———— triangle is a triangle having two equal ————. § 14.

8. If two adjacent angles are ————, their exterior sides lie in a straight line. § 51, 6.

9. Two lines in the same plane must either be ———— or they must intersect. § 105.

10. If a line is perpendicular to one of two parallel lines, it is ———— to the other. § 109.

11. If one of the facts in the hypothesis of a theorem is interchanged with one of the facts in the conclusion, the resulting theorem is a ———— of the original. § 74.

12. If the conclusion of a correct line of reasoning is shown to be false, then the hypothesis from which the conclusion follows must be ————. § 102.

TEST SIX

True-False Statements

If a statement is always true, mark it so. If it is not, replace each word in italics by a word which will make the statement always true.

1. The sum of any two sides of a triangle is *greater* than the third side. *Post.* 4.

2. Two triangles are congruent if two sides and the *acute* angle of one are equal, respectively, to two sides and the *acute* angle of the other. § 64.

3. A straight line is determined by *two* points. *Post.* 3.

4. Any point not on the perpendicular bisector of a line segment *is not* equidistant from the extremities of the segment. § 89.

5. If two triangles have their sides, respectively, equal, their corresponding *angles* are equal. § 80.

6. If two lines intersect and form *right* angles, they are perpendicular to each other. § 26.

7. An *equiangular* triangle is isosceles. § 76.

8. We may prove two segments or angles equal by showing that they are corresponding parts of congruent *triangles*. § 58.

9. In an isosceles triangle, the base angles are *supplementary*. § 69.

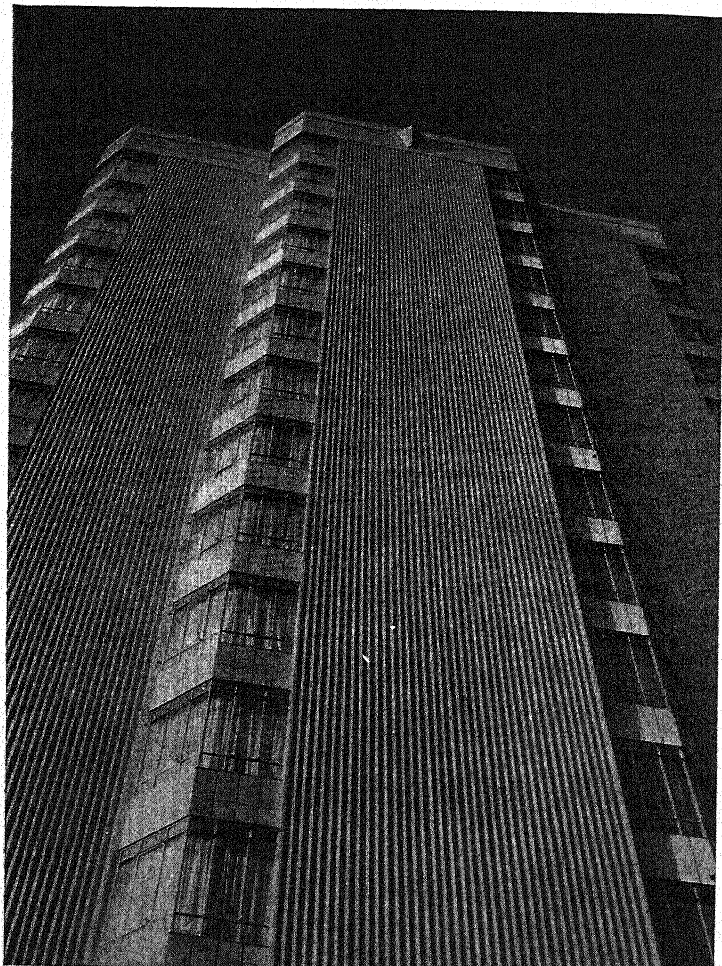
10. In an isosceles triangle the exterior angle formed by one of the equal sides and the base produced is *equal to* one of the opposite interior angles. § 91.

11. Lines perpendicular to the same line are *equal*. § 107.

12. *Three* of the medians of an equilateral triangle are equal. § 79.

13. A line drawn from a vertex of a triangle perpendicular to the opposite side is called a *median*. § 72.

14. A line, no part of which is straight, is called a *broken* line. § 9.



Courtesy Edmond Meany Hotel, Seattle

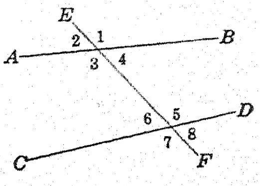
PARALLEL LINES

Vertical parallel lines add to the appearance of height and grace in this modern concrete building.

UNIT THREE

PARALLELS AND THE ANGLES FORMED BY THEM; PARALLELOGRAMS

111. A transversal of lines. A straight line which intersects two or more straight lines is called a **transversal** of those lines. Thus EF is a transversal of AB and CD. Eight angles are formed and are classified as follows: $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$ are called **interior** angles, and $\angle 1$, $\angle 2$, $\angle 7$, and $\angle 8$ are called **exterior** angles.



Two angles such as $\angle 1$ and $\angle 5$, one interior and one exterior, on the same side of the transversal, are called **corresponding** angles.

Two interior angles such as $\angle 3$ and $\angle 5$, on opposite sides of the transversal, are called **alternate interior** angles.

Two angles such as $\angle 2$ and $\angle 8$ are called **alternate exterior** angles. How many pairs of alternate exterior angles are there? Name them.

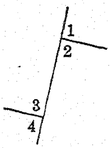
EXERCISES

1. In the figure of § 111, what angles are equal? Why?

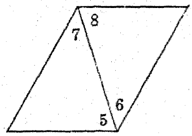
2. What angles in the same figure are supplementary?

3. If $\angle 3$ is 90° and $AB \parallel CD$, how many degrees in each of the other angles? Explain.

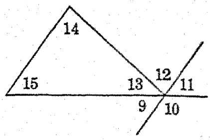
4. Make a large copy of each of the following capital letters: ***A, E, F, H, N, W, Z.*** (*a*) Indicate the transversal in each. (*b*) Indicate the alternate interior angles. (It will be necessary to produce some of the lines.) (*c*) Indicate the corresponding angles.



I



II



III

5. In I, what kind of angles are 1 and 4? 2 and 3?

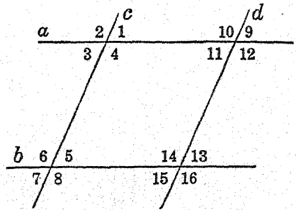
6. In II, what kind of angles are 6 and 7? 5 and 8?

7. In III, what kind of angles are 11 and 15? 12 and 14? 9 and 15?

8. Draw a figure like that in § 111 making $\angle 4 = \angle 6 = 50°$. How large is each of the other angles? What kind of lines do AB and CD seem to be?

9. Repeat Ex. 8 making $\angle 3 = \angle 5 = 70°$.

10. Draw $AB \parallel CD$ and cut by transversal EF. Which of the pairs of angles, named as in § 111, seem to be equal?



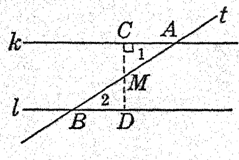
11. If a and b are cut by transversal c, name the alternate interior angles; the corresponding angles.

12. In the same figure, if c and d are cut by transversal a, name the alternate interior angles; the corresponding angles.

*13. In the figure of § 111, if AB is not parallel to CD, and if $\angle 4 = 50°$, can $\angle 6 = 50°$? Why?

*14. Show by the indirect method that, if AB is parallel to CD, a triangle cannot be formed by AB, CD, and a transversal EF.

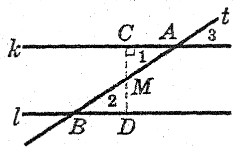
112. Alternate interior angles. If line k is parallel to l, and transversal t intersects k at A and l at B, do you think $\angle 1 = \angle 2$? If AB is bisected at M, and MC is drawn perpendicular to k, and intersects l at D, why is $CD \perp l$?



Why can you not say: "Draw $MC \perp k$ and l"? Do you think the right $\triangle$ ACM and BDM are congruent?

PROPOSITION 1. THEOREM

113. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given: Parallels k and l intersected by transversal t in points A and B.

To prove: $\angle 1 = \angle 2$.

Plan: Bisect AB at M and draw $MC \perp k$. Prove $\triangle ACM \cong \triangle BDM$.

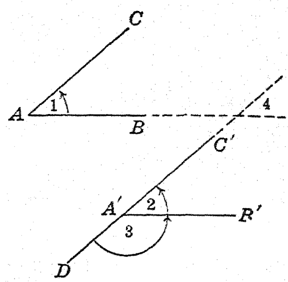
Proof:

STATEMENTS	REASONS
1. From M, the mid-point of AB, draw $MC \perp k$, intersecting l in D. $CD \perp l$.	1. § 109.
2. $\angle AMC = \angle BMD$.	2. § 46.
3. $AM = MB$.	3. *Constructed so.*
4. $\triangle ACM \cong \triangle BDM$.	4. § 98.
5. $\angle 1 = \angle 2$.	5. § 58.

114. COROLLARY 1. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

HINTS. — In the figure of § 113, why is $\angle 1 = \angle 2$? Why is $\angle 2 = \angle 3$? Write the proof in full.

115. COROLLARY 2. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*



116. B. Corollary 3. *If two angles have their sides respectively parallel, they are either equal or supplementary.*

Hint. — Produce the sides of the angles until they intersect, forming $\angle 4$.

$\angle 2 = \angle 4 = \angle 1$. Why?

$\angle 3$ is supplementary to $\angle 2$, hence to $\angle 1$. Why?

117. Initial and terminal sides of an angle. You know that the size of an angle is determined by how much one side must revolve to take the position of the other side. It is convenient to think of one side always revolving in a direction opposite to the hands of a clock, or counterclockwise, as indicated in the illustration. We may, then, call the original position of the side, the **initial** side of the angle, and the final position of the revolving line, the **terminal** side.

In the figure for § 116, AB, the *initial* side of $\angle 1$ is parallel to $A'B'$, the *initial* side of $\angle 2$ and AC, the *terminal* side of $\angle 1$ is parallel to $A'C'$, the *terminal* side of $\angle 2$. Here the angles are *equal.*

In angles 1 and 3: AB, the *initial* side of $\angle 1$ is parallel to $A'B'$, the *terminal* side of $\angle 3$ and AC, the *terminal* side of $\angle 1$ is parallel to $A'D$, the *initial* side of $\angle 3$. Here the angles are *supplementary.*

EXERCISES

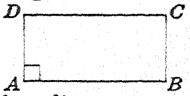
1. Draw a triangle and mark with an arrow the counterclockwise direction of each angle. Name the initial and terminal sides of each.

2. Draw a quadrilateral and mark with an arrow the counterclockwise direction of each angle. Name the initial and terminal sides of each.

3. In graphical work in algebra you have seen how positive and negative lines are defined. Can you explain how we can define positive and negative angles?

4. If, in the figure of § 113, $k \parallel l$ and $\angle 1 = 40°$, find the number of degrees in each of the other seven angles.

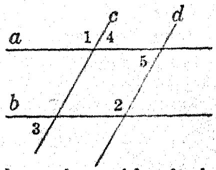
5. Draw a diagram to represent the rails of two railroads which intersect at an angle of 70°. Give the value of each angle.



6. **Given:** $AB \parallel CD$, $AD \parallel BC$, $\angle A = 90°$.

To prove: $\angle$s B, C, and D are 90°.

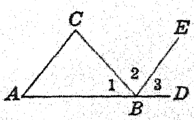
7. In the figure of § 113, if $k \parallel l$ and $\angle 1$ is half the adjacent angle at A, how many degrees in each of the eight angles?



8. If $a \parallel b$ and $c \parallel d$, prove:

(*a*) $\angle 1 = \angle 2$.
(*b*) $\angle 3 = \angle 4$.
(*c*) $\angle 1 + \angle 5 = 180°$.

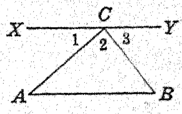
9. Prove that if a line is drawn through a point on one side of a right angle, parallel to the other side, it is perpendicular to the side through which it is drawn.



10. If DBC is an exterior angle of $\triangle ABC$, and $BE \parallel AC$, prove that $\angle DBC$ equals $\angle A + \angle C$.

11. In the same figure, prove $\angle A + \angle B + \angle C = 180°$.

HINT. — $\angle 1 + \angle 2 + \angle 3 = ?$ $\angle 2 = ?$ $\angle 3 = ?$



12. If $XY \parallel AB$, prove $\angle A + \angle B + \angle C = 180°$.

13. If a transversal cuts two parallel lines, making one of two consecutive interior angles equal to three times the other, how many degrees in each of the eight angles?

14. A line parallel to the base of an isosceles triangle and intersecting the equal sides, forms another isosceles triangle.

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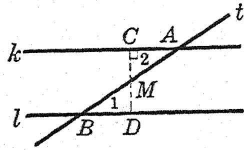
15. If a line is drawn through any point in the bisector of an angle, parallel to either side of the angle, an isosceles triangle is formed.

Proposition 2. Theorem

118. *If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.*



Given: k and l cut by transveral t in points A and B, so that $\angle 1 = \angle 2$.

To prove: $k \parallel l$.

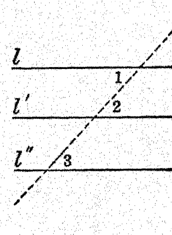
Plan: Bisect AB and draw $MC \perp k$. Then prove that the triangles so formed are congruent, and that each is therefore a right triangle.

Proof: Write the proof in full.

119. Corollary 1. *If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.*

120. Corollary 2. *If two lines are cut by a transversal so that two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

121. Corollary 3. *Two lines parallel to a third line are parallel to each other.*



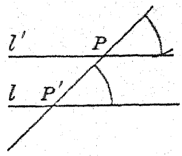
Given: $l' \parallel l$, $l'' \parallel l$.

To prove: $l' \parallel l''$.

Plan: Draw transversal t. Then $\angle 1 = \angle 2$ and $\angle 1 = 3$. (Why?) Hence $\angle 2 = \angle 3$ and $l' \parallel l''$. (Why?)

Construction VI

122. *To construct a parallel to a given line through a given point.*



Given: Line l and point P.

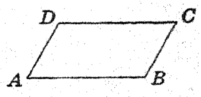
Required: Through P construct a line parallel to l.

Construction: 1. Through P draw any line cutting l at P'.

2. At P construct an angle equal to $\angle P'$. Let l' be a side of this angle. l' is the required line.

Proof: Use § 119.

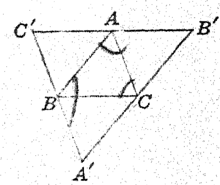
EXERCISES



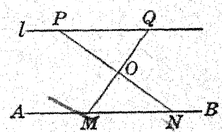
1. Construct quadrilateral $ABCD$ as follows: $AB = 2$ in., $\angle A = 45°$, $AD = 1$ in., $\angle D = 135°$, $DC = 2$ in., and $\angle C = 45°$. Show that $AB \parallel DC$ and $AD \parallel BC$.

Hint. — To construct an angle of 45°, construct a right angle and bisect it. $135° = 90° + 45°$.

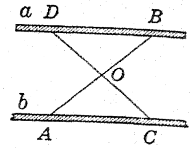
2. Construct quadrilateral $ABCD$ as follows: $AB = 2$ in., $\angle A = 45°$, $AD = 1$ in., $DC \parallel AB$, $BC \parallel AD$. How large are angles B, C, and D?



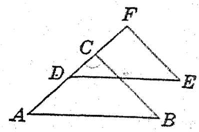
3. Draw any triangle ABC. Through A, B, and C construct parallels to BC, AC, and AB, respectively. If $\angle A = 60°$, $\angle B = 50°$, and $\angle C = 70°$, find $\angle$s A', B', and C'.



4. To determine a line l through P parallel to a given line AB a surveyor proceeds as follows: He measures off a line from P to a point N in AB, and locates O, the middle point of this line. Then from M, a second point in AB, he runs a line through O, and extends it to a point Q so that $OQ = MO$. Show that $QP \parallel AB$.



5. If a represents the top of an ironing board, and b the floor, and if $AO = OB$, $CO = OD$, prove that a is always $\parallel$ b.



6. **Given:** $AB \parallel DE$, $BC \parallel EF$.
To prove: $\angle B = \angle E$.

7. A small triangle $A'B'C'$ is situated inside a larger one, ABC, with $A'B' \parallel AB$, $B'C' \parallel BC$, $C'A' \parallel CA$. Prove that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

8. In the figure for Ex. 6, if $AB \parallel DE$, $BC \parallel EF$, $AD = CF$, prove that $AB = DE$.

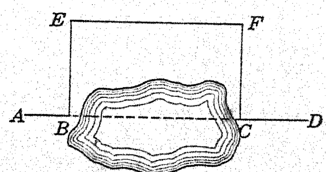
---

9. If the consecutive angles of a quadrilateral are supplementary its opposite sides are parallel.

10. If two parallel lines are intersected by a transversal what relation exists between the bisectors of the corresponding angles? Prove it.

11. Prove that if two parallel lines are intersected by a transversal, the bisectors of the alternate interior angles are parallel.

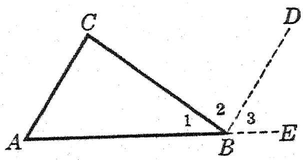
*12. In surveying, a method of extending a line AB beyond an obstacle is as follows: A line BE is run perpendicular to AB, then EF is run perpendicular to BE, then FC is run perpendicular to EF and equal to BE, then CD is run perpendicular to FC. Prove that CD and AB lie in one straight line.



*13. In Ex. 3, prove that the altitudes of $\triangle ABC$ are the perpendicular bisectors of the sides of $\triangle A'B'C'$.

Proposition 3. Theorem

123. *The sum of the angles of a triangle is equal to a straight angle.*



Given: $\triangle ABC$.

To prove: $\angle A + \angle B + \angle C = 1 \text{ st. } \angle$.

Plan: Form the exterior angle at B and draw $BD \parallel AC$. Use parallel line theorems.

Proof:

STATEMENTS	REASONS
1. Produce AB to E and draw $BD \parallel AC$.	1. § 122.
2. $\angle 2 = \angle C$, $\angle 3 = \angle A$.	2. §§ 113, 114.
3. $\angle 1 + \angle 2 + \angle 3 = 1 \text{ st. } \angle$.	3. § 27.
4. $\angle A + \angle B + \angle C = 1 \text{ st. } \angle$.	4. *Ax.* 7.

124. Corollary. 1 *A triangle can have but one right angle or one obtuse angle.*

Suggestion. — Suppose it had two right angles or two obtuse angles; then apply § 123.

125. Corollary 2. *The acute angles of a right triangle are complementary.*

126. Corollary 3. *If two angles of one triangle are equal, respectively, to two angles of another triangle, the third angles are equal.*

127. Corollary 4. *If two triangles have a side, an adjacent angle, and the opposite angle of the one equal, respectively, to the corresponding parts of the other, the triangles are congruent.*

128. Corollary 5. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

EXERCISES

1-3. Prove the theorem in § 123 by the methods suggested by the following figures. In Fig. 1, $XA \parallel BC$; in Fig. 2, $PQ \parallel AB$, $PR \parallel AC$; in Fig. 3, $l \parallel m \parallel n$.

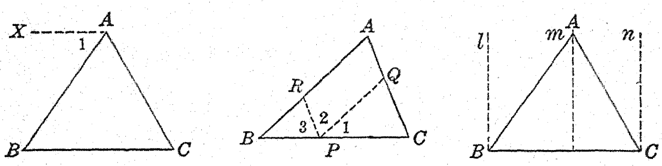


Fig. 1 Fig. 2 Fig. 3

4. If the three angles of a triangle are equal, show that each contains 60°.

5. If one angle of a triangle is 38° 40′, and the other two angles are equal, find the size of each of the equal angles.

6. If one angle of a triangle is 64° 25′ and another is 45° 30′ 30″, find the third angle.

7. Prove that if the angle at the vertex of an isosceles triangle is 60°, each of the other angles is 60°.

8. One of the angles at the base of an isosceles triangle is 45°. Find the size of each of the other angles.

9. The vertex angle of an isosceles triangle is 70°. How large is each base angle?

10. How large is each acute angle in an isosceles right triangle?

11. One of the angles at the base of an isosceles triangle is 37° 35′. Find the size of each of the other angles.

12. Of the three angles of a triangle the second is twice as large as the first, and the third is three times as large as the first. How many degrees are there in each? (Form an equation.)

13. If one angle of a triangle is a right angle, and one of the acute angles is four times the other, how many degrees are there in each?

14. Construct an equilateral triangle and thus construct an angle of 60°.

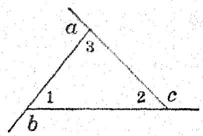
15. Construct an angle of 30°.

---

16. The angle at the vertex of an isosceles triangle is three times as large as either of the other angles. Find by an equation the number of degrees in each angle of the triangle.

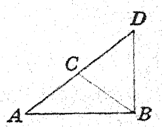
17. The angle at the vertex of an isosceles triangle is 20° more than the sum of the other two angles. Find by an equation the number of degrees in each angle.

18. Given two angles of a triangle, with compasses and straight edge construct the third angle. Base the construction on § 123.



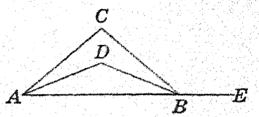
19. In this figure, $\angle a$ equals the sum of what two angles? $\angle b$? $\angle c$? From these equations find the number of degrees in $\angle a + \angle b + \angle c$.

***20.** If one of the equal sides of an isosceles triangle be produced through the vertex its own length, the straight line joining its extremity to the nearer extremity of the base is perpendicular to the base.



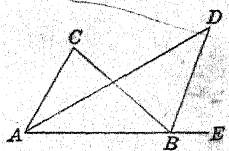
SUGGESTION. — Show that $\angle DBA$ = one half of the sum of the angles of $\triangle ABD$.

***21.** The angle made by the bisectors of the base angles of an isosceles triangle is equal to an exterior angle at the base. (Prove $\angle ADB = \angle EBC$.)



SUGGESTION. — Show that each angle is a supplement of $\angle ABC$.

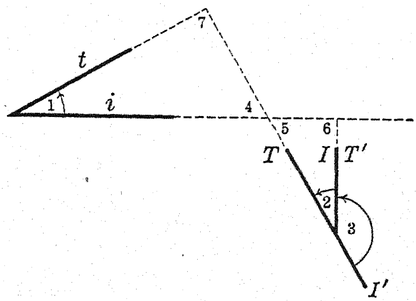
***22.** Prove that if AD bisects $\angle A$ of $\triangle ABC$, and BD bisects $\angle EBC$, the exterior angle, then $\angle D = \frac{1}{2} \angle C$.



SUGGESTION. — Express the sum of the angles of $\triangle ABD$ in terms of $\angle D$, $\angle A$, $\angle B$, and $\angle C$. Set this equal to $\angle A + \angle B + \angle C$. Why?

Proposition 4. Theorem

129. B. *If two angles have their sides, respectively, perpendicular, they are either equal or supplementary.*



Given: $\measuredangle$ 1 and 2 with initial sides i and I, and terminal sides t and T, respectively: $i \perp I$ and $t \perp T$. Also,

Given: $\angle 3$ with initial side $I' \perp t$ and terminal side $T' \perp i$.

To prove: $\angle 1 = \angle 2$ and $\angle 1 + \angle 3 = 1$ st. $\angle$.

Plan: Compare $\measuredangle$ 4 and 5, and $\measuredangle$ 6 and 7. Then use § 126.

Proof:

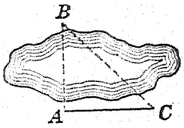
STATEMENTS	REASONS
1. Produce the sides of the angles until they meet: $\angle 4 = \angle 5$.	1. § 46.
2. $\angle 6 = \angle 7$.	2. *Post.* 9.
3. $\therefore \angle 1 = \angle 2$.	3. § 126.
4. $\angle 2$ is supp. to $\angle 3$.	4. § 38.
5. $\therefore \angle 1$ is supp. to $\angle 3$.	5. § 40.

Discussion: Restate the theorem (§ 129), telling when the angles are equal and when supplementary. Also prove the theorem using different figures.

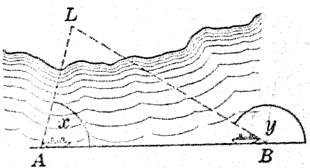
PRACTICAL APPLICATIONS

(OPTIONAL)

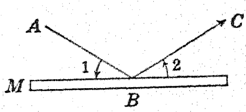
1. In order to determine the distance from A to the inaccessible point B, a line AC is measured off at right angles to AB, and a point C found in this line at which $\angle BCA = 45°$. Prove that $AB = AC$.



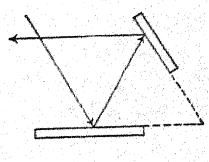
2. The distance BL from his ship to a lighthouse L may be determined by a sailor as follows: When the ship is at any point A, he observes $\angle x$, the angle which L makes with the course AB of the ship. Then he observes when L makes an angle just twice as large with the course of the ship, i.e. when $\angle y = 2\angle x$. From the ship's log he knows how far the ship has traveled from A to B. Prove that this equals the distance from B to L.



3. It is a law of physics that when a ray of light strikes a plane mirror, it is reflected in such a direction that the striking ray and reflected ray make equal angles with the mirror. Thus, if M is an edge view of the mirror and a ray AB is reflected in the direction BC, $\angle 1 = \angle 2$.

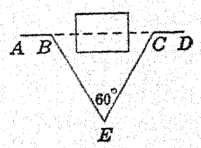


If a ray of light strikes first one mirror and then another, such that the path of the ray forms a triangle with all of its angles equal, what is the angle between the mirrors?



SUGGESTIONS. — How many degrees are there in each angle of the triangle?

4. In running a straight line AB, surveyors ran into an obstacle. Prove that they may pass the obstacle and continue the line CD in line with AB as follows: Measure off an angle of 60° at B, and run the line BE sufficiently long to clear the obstacle. At E construct an angle of 60° as in the diagram, and measure off $EC = BE$. Then at C turn off an angle of 60°, and establish the line CD. CD is a prolongation of AB.



130. A **polygon** is a closed plane figure having three or more sides. The polygon is **convex** if each of its angles is less than a straight angle. Otherwise it is **concave.** The **vertices** are the intersections of the sides. Only convex polygons are studied in this course.

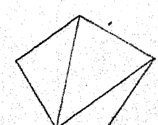
A **diagonal** is a line joining two non-consecutive vertices of the polygon.

131. A polygon of three sides is a **triangle;** of four sides a **quadrilateral;** of five sides a **pentagon.** The others in order are **hexagon, heptagon, octagon, nonagon, decagon,** etc.

Ex. 1. Draw each of the eight figures named.

Ex. 2. Draw a quadrilateral $ABCD$ and a diagonal from vertex A. How many triangles are formed? How is the sum of the angles of the triangles related to the sum of the angles of the quadrilateral? What is the sum of the angles of a quadrilateral?

132. The sum of the angles of a polygon. How many diagonals can you draw from one vertex in a quadrilateral? How many in a pentagon? A hexagon? An octagon?



Into how many triangles is the quadrilateral so divided? The pentagon? The hexagon? The octagon?

If a polygon has 10 sides, it can be divided into —— triangles by diagonals from one vertex.

If a polygon has n sides, it can be thus divided into $n - 2$ triangles.

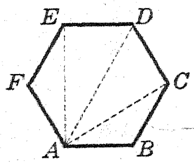
Do the angles of these triangles make up all the angles of the polygon? What is the sum of the angles of a triangle?

Then what is the sum of the angles of a quadrilateral? Of a pentagon? Of a hexagon? Of a polygon having n sides?

See if you can prove the next theorem.

Proposition 5. Theorem

133. *The sum of the angles of a polygon of n sides is $(n-2)$ straight angles.*



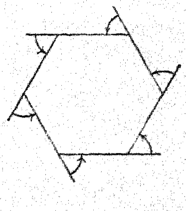
Given: Polygon $ABCDEF$ with n sides.

To prove: The sum of the $\measuredangle s = (n-2)$ st. $\measuredangle s$.

Plan: Divide the polygon into triangles by diagonals from any vertex. There will be a triangle for each side of the polygon except the two sides adjacent to the vertex selected.

Proof: Write the proof in full.

134. Corollary. *The sum of the exterior angles of a polygon made by producing each of the sides in succession is two straight angles.*



Suggestion. — If the polygon has n sides, there are n straight angles in the sum of the interior and exterior angles, since there is one straight angle at each vertex. What is the sum of the interior angles? Then what is the sum of the exterior angles?

135. A **regular polygon** is a polygon whose sides are all equal and whose angles are all equal.

EXERCISES

1. What is the sum of the angles of a quadrilateral? Of a pentagon? Of a hexagon? Of an octagon? Of a decagon? Of a polygon with 12 sides? Of a polygon with 22 sides?

2. How many degrees has each angle of a regular pentagon? Of a regular hexagon? Of a regular octagon? Of a regular decagon?

3. Show that each angle of a regular polygon of n sides equals $\frac{n-2}{n}180°$.

4. The angles of a hexagon are x, x, $1\frac{1}{2}x$, $1\frac{1}{2}x$, $2x$, and $2x$ degrees. How many degrees are there in each?

5. If two angles of a quadrilateral are supplementary, show that the other two angles must be supplementary also.

6. How many degrees in each exterior angle of a regular hexagon? Of a regular octagon? Of a regular decagon?

7. How many sides has a polygon the sum of whose angles is 14 right angles?

8. How many sides has a regular polygon each of whose angles is 150°?

HINT. — How large is each exterior angle?

9. How many sides has a regular polygon if each angle is 108°?

10. How many sides has a regular polygon each of whose angles is $1\frac{3}{7}$ right angles?

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11. If a line which bisects an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.

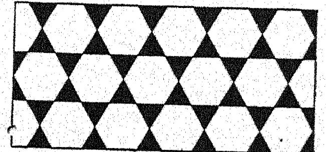
12. The equal sides of an isosceles triangle are produced through the vertex and a line is drawn intersecting these sides produced, and parallel to the base of the triangle. Prove that the triangle so formed is isosceles.

*13. Prove the theorem in § 133 by drawing lines from any point O within the polygon to all the vertices.

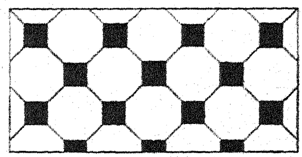
SUGGESTION. — What angles must be subtracted from the sum of the angles of the n triangles to obtain the sum of the angles of the polygon?

PRACTICAL APPLICATIONS

(OPTIONAL)

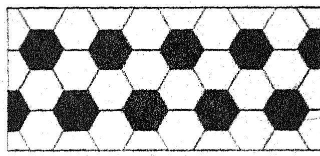


1. In the adjoining figure the tiles of the floor are all regular hexagons and equilateral triangles. Will two such hexagons and two such triangles completely cover the angle about a point? Explain.



2. In this linoleum pattern, only regular octagons and squares are used. Prove that two such octagons and a square completely cover the angle about a point.

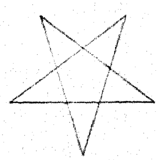
3. Make drawings of other tile floors and linoleum patterns, of parquet flooring, etc., which you have seen, and show how regular polygons are employed in designing them.

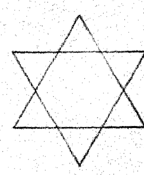


4. Show that a floor can be laid by use of tiles in the form of regular hexagons alone. How many such tiles may be placed so as to completely cover the angle about a point?

5. Tiles of the forms of what other regular polygons may be used alone for laying floors? Explain why. Why cannot a floor be laid entirely of tiles that are regular pentagons? Regular octagons?

6. What advantage has the bee in always building the cells of its comb in the form of a regular hexagon? Is there any waste space?

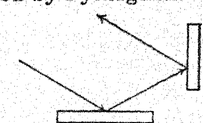




7. How may a six-pointed star be constructed from a regular hexagon? How many degrees in each point of the star?

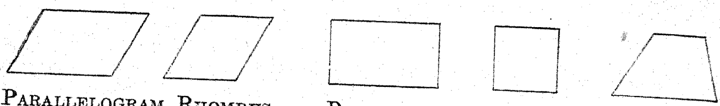
8. How may a five-pointed star be constructed from a regular pentagon? How many degrees in each point of the star?

NOTE. — Star polygons, such as those in Ex. 7 and Ex. 8, are used extensively in modern ornament. Almost every piece of cut glass contains one or more of them. The five-pointed star in Ex. 8 is the one used in the American flag. The use of star polygons dates back to ancient times. The five-pointed star was used as a symbol of recognition by the members of the Greek school or secret society founded by Pythagoras.



9. If a ray of light is reflected first by one mirror and then by another, so that it leaves the second mirror along a line parallel to its path before it struck the first one, what is the angle between the mirrors? (See Ex. 3, page 131.)

Special Quadrilaterals



Parallelogram Rhombus Rectangle Square Trapezoid

136. A **parallelogram** (▱) is a quadrilateral whose opposite sides are parallel.

Any side of a parallelogram may be taken as its **base.**

The **altitude** of a parallelogram is the segment perpendicular to the base from any point in the opposite side.

137. A **rhombus** is a parallelogram having two adjacent sides equal.

In general, the angles of a rhombus are not right angles. We shall prove that all the sides of a rhombus are equal.

138. A **rectangle** is a parallelogram one of whose angles is a right angle.

From the definition of a parallelogram and the relation between the interior angles on the same side of a transversal cutting two parallel lines, what can you say about the other angles?

139. A **square** is a rectangle having two adjacent sides equal.

We shall prove that a square has all its sides equal.

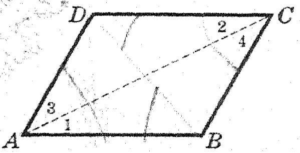
140. A **trapezoid** is a quadrilateral having two and only two of its sides parallel.

The parallel sides of a trapezoid are its **bases.**

The **altitude** of a trapezoid is the segment perpendicular to one base from any point in the other.

PROPOSITION 6. THEOREM

141. *The opposite sides of a parallelogram are equal and the opposite angles are equal.*



Given: $\square ABCD$.

To prove: $AB = DC$, $AD = BC$, $\angle A = \angle C$, $\angle B = \angle D$.

Plan: Draw a diagonal and prove that the triangles are congruent.

Proof: The proof is left for you to work out.

142. COROLLARY 1. *A parallelogram is divided into two congruent triangles by either diagonal.*

143. COROLLARY 2. *Segments of parallel lines cut off by parallel lines are equal.*

144. Distance between parallel lines. The distance between two parallel lines is the length of the perpendicular from any point in one of the lines to the other.

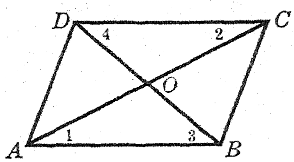
145. COROLLARY 3. *Two parallel lines are everywhere equidistant.*

146. COROLLARY 4. *If the opposite angles of a quadrilateral are equal, it is a parallelogram.*

HINT. — How can you prove that a quadrilateral is a $\square$ (§ 136)? How do you prove lines are $\parallel$ (§§ 118–120)?

PROPOSITION 7. THEOREM

147. *The diagonals of a parallelogram bisect each other.*



Given: ▱ $ABCD$, diagonals AC and BD intersecting at O.

To prove: $AO = OC$ and $BO = OD$.

Plan: Prove the triangles congruent.

Proof: Left for you to give.

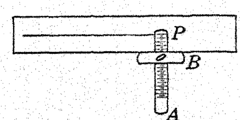
HINT. — Why is $\angle 1 = \angle 2$? $\angle 3 = \angle 4$?

148. COROLLARY. *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

SUGGESTION. — Since the triangles formed are congruent, what angles are equal? Then why are the opposite sides parallel?

Ex. 1. Each diagonal of a rhombus is the perpendicular bisector of the other.

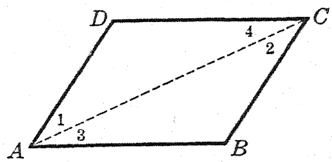
Ex. 2. Carpenters use a tool called a gauge for marking a line parallel to the edge of a board. The part A carries a marking point P. The part B may be adjusted on the part A at any required distance from point P by means of a thumb-screw. By placing the tool as shown in the figure, with the part B against the edge of a board, and moving the gauge, the point P marks a line on the board parallel to the edge. Why?



Ex. 3. A line through the intersection of the equal sides of an isosceles triangle, parallel to the base, bisects the exterior angle at that vertex.

PROPOSITION 8. THEOREM

149. *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*



Given: Quadrilateral $ABCD$, $AB = DC$, $AD = BC$.

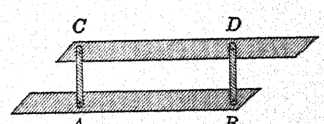
To prove: $ABCD$ is a $\square$.

Plan: Prove the $\triangle$s congruent, and thus show that the alternate interior angles are equal.

Proof:

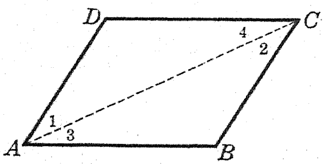
STATEMENTS	REASONS
1. Draw AC. $AD = BC, AB = DC$.	1. *Given.*
2. $AC = AC$.	2. *Same line.*
3. $\triangle ADC \cong \triangle ABC$.	3. s.s.s. = s.s.s.
4. $\angle 1 = \angle 2, \angle 3 = \angle 4$.	4. § 58.
5. $AD \parallel BC, AB \parallel DC$.	5. § 118.
6. $ABCD$ is a parallelogram.	6. § 136.

Ex. 1. The figure shows an instrument called a parallel ruler, used for drawing parallel lines. It consists of two rulers, AB and CD, which are connected by two cross-pieces, AC and BD. The four parts work on pivots at A, B, C, and D, so that by revolving AC and BD the rulers may be brought closer together or placed farther apart. $AB = DC$ and $AC = BD$. Prove that for all positions, $AB \parallel CD$. Explain how the instrument may be used for drawing parallel lines.



Proposition 9. Theorem

150. *If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.*



Given: Quadrilateral $ABCD$, $AD = BC$, $AD \parallel BC$.

To prove: $ABCD$ is a $\square$.

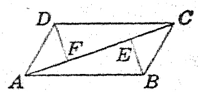
Plan: Using congruent $\triangle$ show that the alternate interior angles are equal.

Proof:

STATEMENTS	REASONS
1. Draw AC. $AD = BC$, and $AD \parallel BC$.	1. *Given.*
2. Hence $\angle 1 = \angle 2$.	2. § 113.
3. $AC = AC$.	3. *The same line.*
4. $\triangle ADC \cong \triangle ABC$.	4. *s.a.s.* = *s.a.s.*
5. $\angle 3 = \angle 4$.	5. § 58.
6. $AB \parallel CD$.	6. § 118.
7. $\therefore ABCD$ is a parallelogram.	7. § 136.

EXERCISES

1. The consecutive angles of a parallelogram are supplementary.
2. All the sides of a rhombus are equal.
3. All the sides of a square are equal.
4. Each angle of a rectangle is 90°.
5. Each angle of a square is 90°.
6. If the diagonals of a rectangle are perpendicular to each other, the rectangle is equilateral.



7. $ABCD$ is a parallelogram and BE and DF are perpendicular to AC. Prove $BE = DF$.

8. A line drawn through the point of intersection, O, of the diagonals of a parallelogram, and terminated by two opposite sides, is bisected by O.

9. The line joining the mid-points of two opposite sides of a parallelogram is parallel and equal to the other sides.

10. If $ABCD$ is a parallelogram, and E and F the middle points of DC and AB, respectively, then $BEDF$ is also a parallelogram.

11. If the angle at the vertex of an isosceles triangle is equal to one half of a base angle, the bisector of a base angle divides the triangle into two isosceles triangles. (Let the vertex angle equal x.)

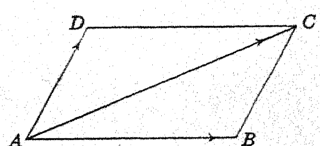
12. If either diagonal of a parallelogram bisects one of the angles, the parallelogram is equilateral.

***13.** If from any point in the base of an isosceles triangle parallels to the sides are drawn, the parallelogram thus formed has the same perimeter for all positions of the point.

PRACTICAL APPLICATIONS

(OPTIONAL)

1. If two forces are exerted in different directions upon the same object at A, they have the same effect as a single force called their resultant. If the directions and magnitudes of the two forces are represented by the line segments AB and AD, the direction and magnitude of the resultant will be represented by the line segment AC, diagonal of the parallelogram $ABCD$.



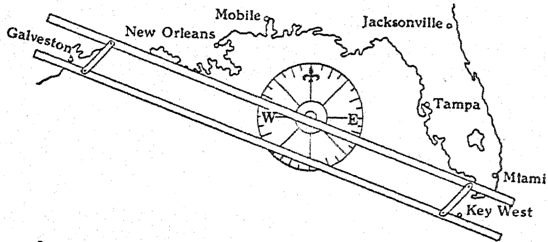
A force of 100 lb. and another of 200 lb. are exerted upon an object at an angle of 45° with each other. Representing 100 lb. by a line segment 2 in. long, draw the forces to scale and find the resultant. (Measure diagonal with ruler and compute resultant in pounds.)

In the same way, compute the resultant force if $\angle BAD = 60°$.

2. Two forces are exerted upon an object at right angles with each other. One force is 400 lb. and the other 600 lb. Construct and compute their resultant as in Ex. 1.

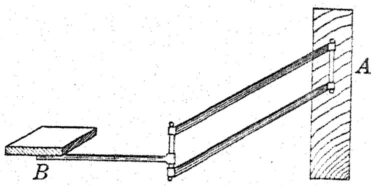
3. The parallel ruler is used by sailors in determining the courses

of their ships in sailing from one port to another. Thus, to determine the course or direction from Galveston to Key West, in the Gulf

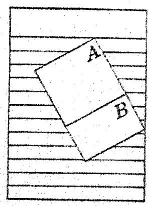


of Mexico, the parallel ruler is placed so that one ruler connects these two points on a navigator's map. Explain how the other ruler may then be placed so as to read the required direction on the mariner's compass which is printed on the map.

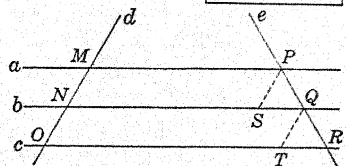
4. An adjustable bracket is fastened to the wall at A, and carries a shelf B. Explain why the shelf B remains horizontal in all positions when the shelf is raised and lowered.



151. Segments intercepted by parallels. Mark a segment $AB = 1\frac{3}{4}$ in. on the edge of your paper, and place it on a piece of ruled paper (with the parallel lines equally spaced), so that the first line passes through A and the fifth line through B. Into how many equal parts is AB divided?



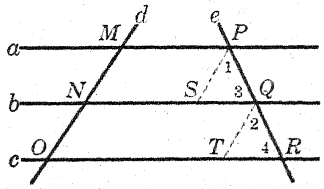
In the next figure, if a, b, and c are parallel, and are intersected by transversals d and e, and if PS and QT are drawn parallel to d, can you prove $\angle SPQ = \angle TQR$, and $\angle PQS = \angle QRT$? If $PQ = QR$, why are the triangles congruent? Why is $MN = NO$? Prove that



If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.

Proposition 10. Theorem

152. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given: Parallels a, b, c, cut by transversals d and e in the points M, N, O, and P, Q, R, respectively, so that $PQ = QR$.

To prove: $MN = NO$.

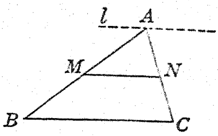
Plan: 1. Draw $\parallel$s and prove the $\triangle$s formed are congruent. 2. Then prove NP and OQ are ▱s.

Proof:

STATEMENTS	REASONS
1. Suppose PS and $QT \parallel d$. Then in $\triangle$s PQS and QRT, $PQ = QR$.	1. *Given.*
2. $PS \parallel QT$.	2. § 121.
3. $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.	3. § 114.
4. $\triangle PQS \cong \triangle QRT$.	4. *a.s.a.* = *a.s.a.*
5. $PS = QT$.	5. § 58.
6. NP and OQ are ▱s.	6. § 136.
7. $PS = MN, QT = NO$.	7. § 141.
8. $\therefore MN = NO$.	8. *Ax.* 1.

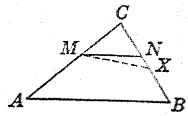
***Ex. 1.** Prove the theorem (§ 152) in two other ways: (1) by drawing lines through M and N parallel to e; (2) by drawing a line through $N \parallel e$.

153. Corollary 1. *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.*



Hint. — If l is $\parallel$ BC, can you apply § 152?

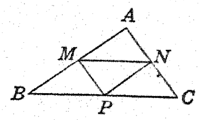
154. Corollary 2. *If a line connects the mid-points of two sides of a triangle, it is parallel to the third side.*



Suggestion. — Use the indirect method.

EXERCISES

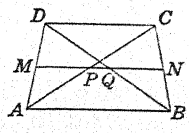
1. If lines are drawn connecting the mid-points of the sides of a triangle, they divide the given triangle into four congruent triangles.



Hint. — Why is $MNPB$ a $\square$? Why is $MNCP$? Then use § 142.

2. In Ex. 1, if $BC = 12$ in., how long is MN? If the perimeter of $\triangle ABC$ is 34 in., what is the perimeter of $\triangle MPN$?

3. In the trapezoid $ABCD$, M, P, Q, and N are the mid-points, respectively, of AD, AC, BD, and BC. Why is $MP \parallel DC$? Why is $QN \parallel DC$? Why is $PN \parallel AB$?



4. In Ex. 3, if $MP \parallel DC$, and $AB \parallel DC$, why is $MP \parallel AB$?

5. If $MP \parallel AB$ and $PN \parallel AB$, why is MPN a straight line?

---

6. Prove that a line through the mid-point of one side of a trapezoid, parallel to the bases, bisects the other side also.

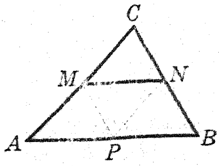
7. Prove that the line connecting the mid-points of the non-parallel sides of a trapezoid is parallel to the bases.

Suggestion. — Use the indirect method. Assume that it is not parallel to the bases and draw a line through the mid-point of one side parallel to the base. Show, by Ex. 6, that this line coincides with the first line.

8. Prove that a line through the mid-point of one side of a trapezoid, parallel to the bases, bisects the diagonals of the trapezoid.

Proposition 11. Theorem

155. *A line segment connecting the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*



Given: $\triangle ABC$, MN bisects AC and BC.

To prove: $MN = \frac{1}{2} AB$ and $MN \parallel AB$.

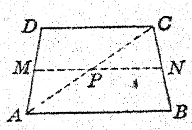
Plan: Let P bisect AB. Draw MP, NP. Then $MNPA$ and $MNBP$ are parallelograms (§ 154) and $MN = AP = PB$.

Write the proof in full.

156. **The median** of a trapezoid is a line connecting the mid-points of the non-parallel sides.

An **isosceles trapezoid** is a trapezoid whose non-parallel sides are equal.

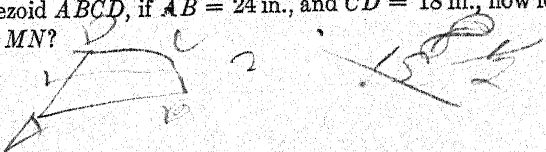
157. Corollary 1. *The median of a trapezoid is parallel to the bases and equal to half their sum.*

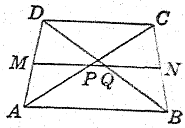


Suggestion. — Draw diagonal AC. Let M, P, and N be the mid-points of AD, AC, and BC, respectively. Draw MP, PN. Then $MP \parallel DC$ (§ 154) and hence to AB. Why? Also $PN \parallel AB$. Why then is MPN a straight line? (§ 106.) Complete the proof by using § 155.

EXERCISES

1. In trapezoid $ABCD$, if $AB = 24$ in., and $CD = 18$ in., how long is the median MN?

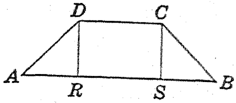




2. In trapezoid $ABCD$, MN is the median and intersects AC at P and BD at Q. If $CD = 18$ in., and $AB = 24$ in., how long is MP? QN? PQ?

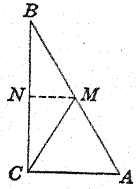
3. Using the same figure, if $AB = 16$ in., and $CD = 8$ in., find MN, MP, QN, and PQ.

4. In the same figure, if $MN = 11$ in., and $AB = 14$ in., find CD, MQ, QN, and PQ.



5. In isosceles trapezoid $ABCD$, $AD = CB$, DR and CS are perpendicular to AB, and $\angle A = \angle B = 45°$. If $AB = 20$ in., and $DR = 5$ in., how long is DC?

6. In the isosceles trapezoid $ABCD$, if $AB = 12$ in., $\angle A = \angle B = 45°$, and $DR = 3$ in., how long is the median?

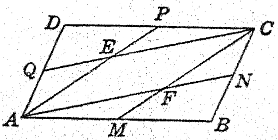


7. In $\triangle ABC$, CM is a median and $\angle ACB = 90°$. Prove that $CM = \frac{1}{2} AB$.

HINT. — Draw $MN \parallel CA$.

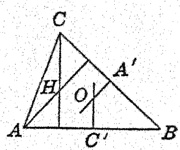
8. Using the fact that in a right triangle the median is equal to half the hypotenuse prove that, if $\angle B = 30°$, $CA = \frac{1}{2} AB$.

9. A perpendicular drawn to the base of an isosceles triangle from the mid-point of one of the equal sides cuts off on the base a segment equal to one-quarter of the base.



10. If $ABCD$ is a parallelogram, M, N, P, and Q the middle points of AB, BC, CD, and AD, respectively, and AP and CQ intersect at E, and AN and CM intersect at F, prove that $AFCE$ is a parallelogram.

11. Prove that the segment connecting the mid-points of the diagonals of a trapezoid is parallel to the bases and equal to half their difference.

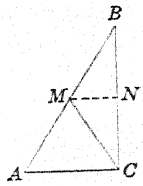


*12. The perpendicular bisectors of two sides of $\triangle ABC$ intersect at O, while the altitudes to the corresponding sides intersect at H. Prove that $OA' = \frac{1}{2} AH$ and $OC' = \frac{1}{2} CH$.

SUGGESTION. — Bisect AH at M and CH at N. Prove $\triangle MHN \cong \triangle A'OC'$.

158. The 30°-60° right triangle. If one acute angle of a right triangle is 30°, the other acute angle, by § 125 is 60°. This triangle is important in the solving of exercises.

159. B. Theorem. *In a right triangle the median to the hypotenuse is equal to half the hypotenuse.* (Prop. 12.)



Given: $\triangle ABC$; $\angle C = 90°$; $AM = BM$.

To prove: $CM = \frac{1}{2} AB$.

Plan: Draw $MN \parallel AC$. $MN \perp BC$. Therefore $\triangle BCM$ is isosceles. Why?

160. B. Corollary 1. *In a 30°–60° right triangle, the hypotenuse is double the side opposite the 30° angle.*

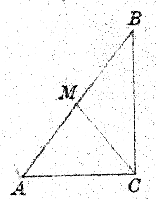
Suggestion. — Draw the median CM. Prove $\triangle ACM$ equilateral.

161. B. Corollary 2. *If the hypotenuse of a right triangle is double one of the sides, then the acute angle opposite that side is 30°, while the other one is 60°.*

Suggestion. — $AC = CM = MB = AM$.

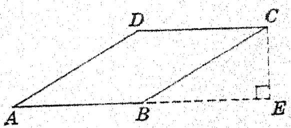
EXERCISES

In $\triangle ABC$, $\angle C = 90°$, $AM = MB$.



1. Find AM if $AC = 20$ in. and $\angle B = 30°$.
2. Find AB if $AC = 12.5$ in. and $\angle A = 60°$.
3. Find AB if $CM = 8$ in.
4. Find CM if $AB = 30$ in.

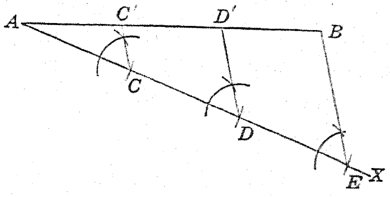
In parallelogram $ABCD$, $CE \perp AB$.



5. Find CE if $AD = 12$ in. and $\angle A = 30°$.
6. Find CE if $AD = 16$ in. and $\angle B = 150°$.
7. Find BE if $CE = 10$ in. and $\angle A = 45°$.
8. Find BE if $AD = 15$ in. and $\angle A = 60°$.

CONSTRUCTION VII

162. *To divide a given line segment into any number of equal parts.*



Given: Segment AB.

Required: To divide AB into three equal parts.

Construction: 1. Draw AX making any convenient angle with AB.

2. With any convenient radius take $AC = CD = DE$. Draw BE.

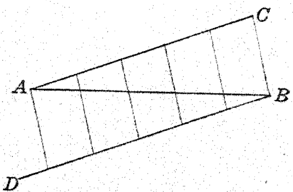
3. Draw through D and C $\parallel$s to BE. Then AB is divided into three equal parts.

Proof: (Draw a line through $A \parallel CC'$.)

STATEMENT	REASON
1. $AC' = C'D' = D'B$.	1. § 152.

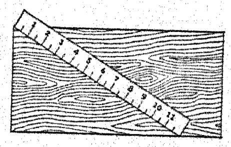
EXERCISES

1. Divide a given line segment into three equal parts; into five equal parts; into eight equal parts.



2. A line segment AB may be divided into any required number of equal parts, say five, as follows: draw AC, making a convenient angle BAC, and draw $BD \parallel AC$. Mark off five equal distances from A on AC and five distances of the same length from B on BD. Connect the points of division as shown in the figure. Prove that AB is divided into five equal parts.

3. Using ruled paper divide a segment $4\frac{1}{2}$ in. long into seven equal parts.



4. A board is to be ripped into three strips of equal width. A carpenter places his 12-inch ruler as shown in the drawing, and

marks points on the board at the 4-inch and 8-inch marks. He then rips the board by sawing through these points parallel to the length of the board. Prove that the strips are of equal width.

5. The diagonals of a square bisect the angles of the square.

6. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

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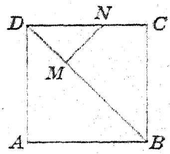
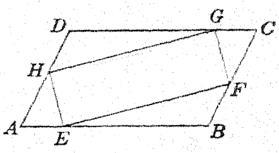
7. $ABCD$ is a square. On the diagonal BD, BM is taken equal to BC, and MN is drawn perpendicular to BD. Prove that $DM = MN = NC$.

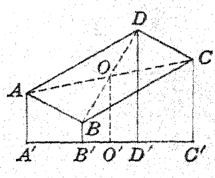


8. In the parallelogram $ABCD$, if E, F, G, and H are so taken on the sides AB, BC, CD, and DA, respectively, that $AE = AH = CF = CG$, prove that $EFGH$ is a parallelogram.

9. In the parallelogram $ABCD$, if points E, F, G, and H are so taken on the sides AB, BC, CD, and DA, respectively, that $AE = BF = CG = DH$, prove that $EFGH$ is a parallelogram.

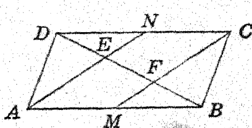


*10. If perpendiculars AA', BB', CC', and DD' are drawn to the line $A'C'$ from the four vertices of ▱ $ABCD$, the sum of the perpendiculars AA' and CC' equals the sum of the perpendiculars BB' and DD'.



SUGGESTION. — What is the relation of $AA' + CC'$ to OO'? What is the relation of $BB' + DD'$ to OO'? (See § 157.)

*11. $ABCD$ is any parallelogram, and M and N are the mid-points of AB and CD, respectively. Prove that AN and CM trisect (cut into three equal parts) DB.



SUGGESTION. — It may be shown that $DE = EF$ by § 153 if it is first proved that $EN \parallel FC$. It may be proved that $EN \parallel FC$ if it is first proved that $AMCN$ is a parallelogram. Hence begin by proving that $AMCN$ is a parallelogram.

Inequalities

163. Axioms relating to inequalities. Two inequalities, such as $5 < 7$ (read *5 is less than 7*) and $4 < 9$, are inequalities in the same order; while the two inequalities $5 < 7$ and $9 > 4$ are in the opposite order.

164. Axiom 8. *If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by the same positive number, the results are unequal in the same order.*

Since $7 > 5$, then $7 + 2 > 5 + 2$; $7 - 4 > 5 - 4$; etc.

165. Axiom 9. *If unequals are subtracted from equals, the results are unequal in the opposite order.*

Thus, since $10 = 10$, and $3 < 8$, then $10 - 3 > 10 - 8$.

166. Axiom 10. *If unequals are added to unequals in the same order, the results are unequal in the same order.*

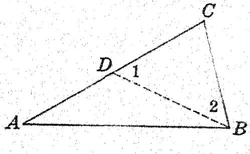
Thus, since $11 > 3$ and $9 > 2$, then $11 + 9 > 3 + 2$.

167. Axiom 11. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

168. In § 91 we proved the inequality:

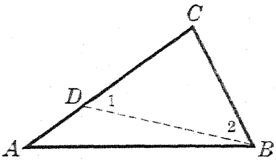
Theorem. *An exterior angle of a triangle is greater than either opposite interior angle.*

169. A triangle with unequal sides. In $\triangle ABC$, if $AC > BC$, on CA we can take $CD = CB$. What kind of triangle is $\triangle CDB$? What angles, then, are equal? $\angle 1$ is an exterior angle of what triangle? What inequality results? How does $\angle B$ compare in size with $\angle 2$? With $\angle A$?



PROPOSITION 13. THEOREM

170. B. *If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.*



Given: $\triangle ABC$ with $AC > BC$.

To prove: $\angle B > \angle A$.

Plan: Take $CD = CB$ and show that $\angle B > \angle 2$, $\angle 2 = \angle 1$, $\angle 1 > \angle A$, and hence that $\angle B > \angle A$.

Proof:

STATEMENTS	REASONS
1. On CA take $CD = CB$. Draw DB. $\angle 1 = \angle 2$.	1. § 69.
2. $\angle B > \angle 2$.	2. *Ax.* 6.
3. Hence $\angle B > \angle 1$.	3. *Ax.* 7.
4. $\angle 1 > \angle A$.	4. § 91.
5. Since $\angle B > \angle 1 > \angle A$, then $\angle B > \angle A$.	5. *Ax.* 11.

171. B. **Unequal angles in a triangle.** Prove the converse of the theorem in § 170 by the indirect method.

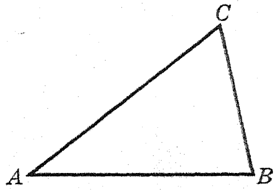
Given: $\angle B > \angle A$.

To prove: $AC > BC$.

Begin: Either $AC > BC$ or AC is not $> BC$. If AC is not $> BC$, there are two possibilities: either $AC = BC$ or $AC < BC$. Consider each of these possibilities in turn and show how each leads to an absurdity.

PROPOSITION 14. THEOREM

172. B. *If two angles of a triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.*



Given: $\triangle ABC$, $\angle B > \angle A$.

To prove: $AC > BC$.

Plan: Use the indirect method of proof.

Proof: Write in full.

Discussion: A direct proof can be given by constructing $\angle XBA = \angle A$ so that X lies on AC. Then $AX = BX$ and $BX + XC > BC$. Hence $AX + XC > BC$, or $AC > BC$.

173. B. COROLLARY 1. *The perpendicular is the shortest segment that can be drawn from a given point to a given line.*

174. B. COROLLARY 2. (Converse of Corollary 1.) *The shortest segment that can be drawn from a given point to a given line is the perpendicular from the point to the line.*

SUGGESTION. — Use the indirect method of proof, applying § 170.

175. In § 99 the **distance from a point to a line** has been defined as the perpendicular distance from the point to the line. Section 174 shows that this distance is the shortest possible distance from the point to the line.

EXERCISES

1. The hypotenuse is greater than either leg of a right triangle.

2. Which is the greatest side of an obtuse triangle?

3. If in $\triangle ABC$, $AC > BC$ and the bisectors of angles A and B meet in D, prove $AD > BD$.

4. If an isosceles triangle is obtuse, the base is the longest side.

5. Side AB of quadrilateral $ABCD$ is greater than BC. Prove that $\angle ACB > \angle CAB$.

6. ABC is an isosceles right triangle with right angle at B. Extend BA its own length through A to D. Prove $\angle BCD > \angle BAC$.

7. In rectangle $ABCD$, $AB > AD$. Prove that diagonal AC does not bisect $\angle s$ A and C.

SUGGESTION. — Use the indirect method of proof.

8. If three sides of a triangle are unequal, all of the angles are unequal.

9. Prove that the sum of two sides of a triangle is greater than the third side.

10. Prove that any side of a triangle is greater than the difference between the other two sides.

11. State in which cases it is possible to draw triangles with sides of the following lengths:

a. 4 in., 5 in., 6 in.
b. 2 in., 4 in., 6 in.
c. 6 in., 12 in., 20 in.
d. $6\frac{1}{2}$ in., $7\frac{1}{2}$ in., 13 in.

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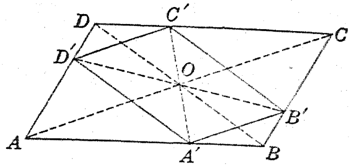
12. The sum of the four sides of any quadrilateral is greater than the sum of the diagonals.

13. The sum of the distances of any point within a triangle to the three vertices is greater than half the sum of the sides.

14. The sum of the distances of any point within any polygon from all of the vertices is greater than one-half of the perimeter.

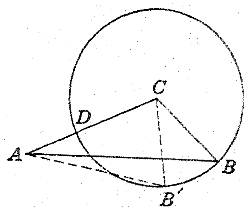
15. In a certain triangle an exterior angle is twice the adjacent interior angle, and the two oppposite interior angles are equal. How many degrees are there in each of the angles of the triangle?

*16. If the vertices of one parallelogram lie upon the four sides of another, the diagonals of both parallelograms intersect at the same point.



SUGGESTION. — Draw the diagonals AC and BD, intersecting at O. Draw OA', OB', OC', OD', and prove $A'OC'$ and $B'OD'$ straight lines, and hence diagonals.

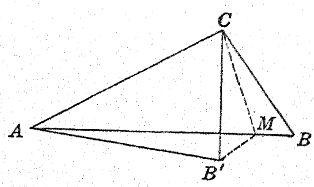
176. Varying the size of an angle. Draw a circle having for its center the vertex C of $\triangle ABC$, and for its radius, side CB. Let the circle intersect CA in D. With this figure you will be able to discover what happens to a side of a triangle as the angle opposite that side changes in size.



Take a point between D and B, such as point B', and draw $B'C$ and $B'A$. What parts of ⊿s ABC and $AB'C$ are equal? How does $\angle ACB$ compare with $\angle ACB'$? What about sides AB and AB'?

Repeat the experiment, taking point B' between B and the point in which AC cuts the circle.

If, while sides AC and BC remain unchanged in size, angle C increases, what happens to side AB? What happens to AB as $\angle C$ decreases?



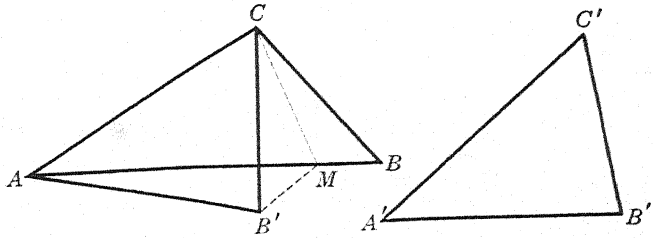
If two triangles, ABC and $A'B'C'$, have $AC = A'C'$, $BC = B'C'$, and $\angle C > \angle C'$, do you think you can show that $AB > A'B'$? Place $\triangle A'B'C'$ on $\triangle ABC$ as in the figure, and let CM bisect $\angle B'CB$. What triangles are congruent?

$AM + MB'$ is greater than what segment?

If you cannot prove that $AB > AB'$, look at the proof in the next section.

PROPOSITION 15. THEOREM

177. B. *If two sides of one triangle are equal, respectively, to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given: $\triangle$ ABC and $A'B'C'$, $AC = A'C'$, $BC = B'C'$, $\angle C > \angle C'$.

To prove: $AB > A'B'$.

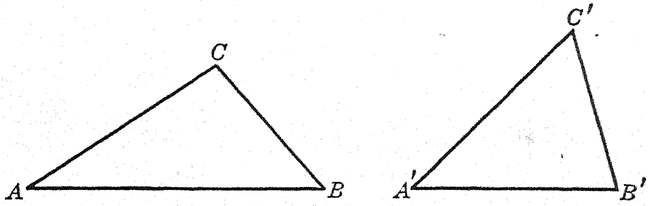
Plan: Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'C'$ coincides with AC. Bisect $\angle B'CB$ and prove $\triangle B'CM \cong \triangle BCM$. Since $AM + MB' > AB'$, and $MB' = MB$, $AB > AB'$.

Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'C'$ coincides with its equal AC. Since $\angle C > \angle C'$, $B'C'$ will fall inside $\angle C$.	1. *Given.*
2. Let CM bisect $\angle B'CB$. Draw $B'M$.	2. *Post.* 10.
3. Now prove $\triangle B'CM \cong \triangle MCB$.	3. *s.a.s.* = *s.a.s.*
4. $AM + MB' > AB'$.	4. *Why?*
5. $AM + MB > AB'$, or $AB > A'B'$.	5. *Why?*

Proposition 16. Theorem

178. B. *If two sides of one triangle are equal, respectivly, to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given: $\triangle ABC$ and $A'B'C'$, $AC = A'C'$, $BC = B'C'$, $AB > A'B'$.

To prove: $\angle C > \angle C'$.

Suggestion. — See if you can prove this by the indirect method. There are two possibilities: Either $\angle C > \angle C'$, or $\angle C$ is not $> \angle C'$. If $\angle C$ is not $> \angle C'$, either $\angle C = \angle C'$ or $\angle C < \angle C'$. If $\angle C = \angle C'$, $\triangle ABC \cong \triangle A'B'C'$ (Why?) and AB would equal $A'B'$.

What is contradicted by the last statement?

Then show that $\angle C$ cannot be $< \angle C'$ (§ 177). Write the proof in full.

EXERCISES

1. Angle A of quadrilateral $ABCD$ is less than 60°. Can diagonal BD be less than AB and also less than AD?

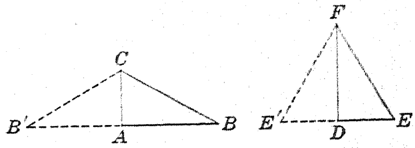
2. Prove that a diagonal of a rectangle is greater than either side of the rectangle.

3. AM is a median of $\triangle ABC$. $\angle AMB$ is acute. Prove that $AC > AM$.

4. Side AB of equilateral triangle ABC is produced through B to D. Prove that $\angle ACD > \angle ADC$.

5. Each leg of an isosceles triangle is greater than half the base.

6. ABC and DEF are right triangles with right $\angle s$ at A and D. If $\angle C > \angle F$, and $BC = EF$, prove $AB > DE$.



SUGGESTIONS. — Produce BA to B', making $AB' = AB$ and draw $B'C$. Also produce ED to E', making $DE' = DE$ and draw $E'F$. Now show that $B'B > E'E$.

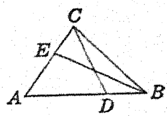
7. $ABCD$ is a rhombus and $\angle A$ is an acute angle. Prove that $AC > BD$.

8. If, in $\triangle ABC$, median AM makes $\angle AMB$ an acute angle, prove that $AC > AB$ and $\angle B > \angle C$.

9. ABC and DEF are right triangles with right angles at A and D. If $BC = EF$ and $AB > DE$, prove $\angle C > \angle F$.

HINT. — See Ex. 6.

10. In $\triangle ABC$, $AB > AC$ and $DB = EC$. Prove $BE > CD$.



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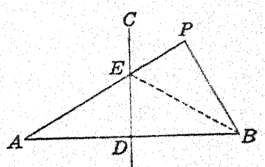
11. If two sides of a triangle are unequal, the median to the third side makes the larger angle with the shorter of the unequal sides.

SUGGESTIONS. — Draw a triangle ABC with $AC > CB$; let CD be the median. Prove that $\angle DCB > \angle ACD$. Produce CD to E, making $CD = DE$; draw AE. Compare $\angle DCB$ with $\angle E$, and $\angle ACD$ with $\angle E$.

12. If two sides of a triangle are unequal, the median to the third side forms an acute angle and an obtuse angle with that side.

13. Using the figure drawn for Ex. 10, prove that $CD < \frac{1}{2}(AB + AC + CB)$.

14. Give a direct proof that any point not on the perpendicular bisector of a line segment is not equidistant from the ends of the segment.



SUGGESTION. — If CD is the perpendicular bisector of AB, and P any point not on CD, let P be on the same side of CD as B, and let AP intersect CD at E. Draw EB. Prove $PB < PA$. (PB is less than the sum of what two line segments?)

179. Summary of the Work of Unit Three. In Unit Three you have learned:

I. ***Two parallel lines intersected by a transversal form:***

1. *Equal corresponding and equal alternate interior angles.*
2. *Interior angles on the same side of the transversal that are supplementary.*

II. ***Two lines are parallel if, when intersected by a transversal, they form:***

1. *Equal corresponding or equal alternate interior angles.*
2. *Interior angles on the same side of the transversal that are supplementary.*

III. ***That in any polygon:***

1. *The sum of the interior angles is $(n - 2)\ 180$ degrees;* and hence
2. *Each interior angle of a regular polygon is* $\frac{(n-2)}{n}\ 180$ *degrees.*
3. *The sum of the exterior angles is 360°.*

IV. ***That in any parallelogram:***

1. *The opposite sides are equal.*
2. *The opposite angles are equal.*
3. *The diagonals bisect each other.*

V. ***That any quadrilateral is a parallelogram if:***

1. *The opposite angles are equal.*
2. *The diagonals bisect each other.*
3. *The opposite sides are equal.*
4. *Two sides are equal and parallel.*

VI. *In special parallelograms:*

1. *The diagonals of a rectangle are equal.*
2. *The diagonals of a rhombus or square are perpendicular to each other.*

VII. ***About inequalities.*** These axioms and theorems are given in §§ 163–178.

VIII. ***Constructions.***

1. *To construct a parallel to a given line through a given point.*
2. *To divide a given segment into any number of equal parts.*

IX. ***New methods of proving original exercises.***

1. *You can prove any two segments are equal by proving that:*
   a. *They are opposite sides of a parallelogram.*
   b. *They are segments intercepted on a transversal by a series of parallels which intercept equal segments on another transversal.*
2. *You can prove any two angles are equal by proving that:*
   a. *They are alternate interior or corresponding angles of parallel lines.*
   b. *They are angles whose sides are, respectively, parallel or perpendicular.*
3. *You can prove any two lines are parallel by proving that:*
   a. *They are parallel to the same line.*
   b. *When intersected by a transversal they form equal alternate interior, or equal corresponding angles.*

c. *When intersected by a transversal, the interior angles on the same side of the transversal are supplementary.*
d. *The lines are opposite sides of a parallelogram.*

4. *You can prove any quadrilateral is a parallelogram by proving that:*
   a. *Its opposite sides are parallel.*
   b. *Its opposite sides are equal.*
   c. *A pair of opposite sides are equal and parallel.*
   d. *A pair of consecutive angles are supplementary.*
   e. *The diagonals bisect each other.*
   f. *The opposite angles are equal.*

5. *You can prove two segments are unequal by proving that:*
   a. *They are opposite unequal angles in a triangle.*
   b. *In two triangles having two sides of one equal, respectively, to two sides of the other, they are the sides opposite the included unequal angles.*

6. *You can prove that two angles are unequal by proving that:*
   a. *They are angles opposite unequal sides in a triangle.*
   b. *They are angles which are opposite unequal sides in two triangles having two sides of one equal, respectively, to two sides of the other.*

REVIEW OF UNIT THREE

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. The references given are those most closely related to the exercise. Then study that section before taking the tests.

1. Define parallel lines; exterior angles; interior angles; corresponding angles; alternate interior angles. §§ 104, 111.

2. Define polygon (concave and convex); vertex; diagonal; pentagon; hexagon; octagon; regular polygon. §§ 130, 131.

3. Define parallelogram; rhombus; rectangle; square. §§ 136–139.

4. Define trapezoid; median of a trapezoid; isosceles trapezoid. §§ 140, 156.

5. Draw two lines cut by a transversal. Number and name the angles formed. § 111.

6. What is meant by the distance between two points? *Post.* 4.

7. What is meant by the distance between two parallel lines? § 144.

8. How many diagonals can be drawn from one vertex in a quadrilateral? In a pentagon? In a hexagon? In an octagon? § 132.

9. Into how many triangles is each of the figures in Ex. 8 divided? § 132.

10. What is the sum of the angles of a triangle? Of a parallelogram? Of a polygon with 10 sides? § 133.

11. What is the sum of the exterior angles of a triangle? Of a quadrilateral? Of a polygon with 12 sides? § 134.

12. Give the two postulates about parallel lines. §§ 105, 106.

13. Give the four axioms about inequalities. §§ 164–167.

14. Give four ways that two lines intersected by a transversal can be proved parallel. §§ 107, 118–120.

15. Give four ways that a quadrilateral can be proved to be a parallelogram. §§ 146, 148–150.

16. Give four methods (learned in this unit) of showing that angles are equal. §§ 113, 114, 116, 129.

Complete the following:

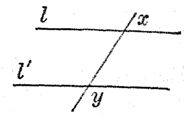
17. The sum of the angles of a polygon of n sides is . . . . § 133.

18. The sum of the exterior angles of a polygon is . . . . § 134.

19. The diagonals of a parallelogram . . . . § 147.

20. If the opposite sides of a quadrilateral are equal . . . . § 149.

NUMERICAL EXERCISES



1. If $x = \frac{2}{3}y$, and $x = 78°$, is $l \parallel l'$? Why?

2. If $y - x = 50°$ and $x = 65°$, is $l \parallel l'$? Why?

3. What must $x + y$ equal to make the lines parallel?

4. One acute angle of a right triangle is 36°. Find the other.

5. One acute angle of a right triangle is $\frac{4}{5}$ the other acute angle. How many degrees in each?

6. One acute angle of a right triangle is three times the other. How many degrees in each?

7. One angle of a parallelogram is two-thirds as large as one of the others. How many degrees in each angle?

8. One angle of a pentagon is twice as large as a second angle, and half as large as a third angle. It contains 36° more than a fourth angle. If the fifth angle is 45°, how many degrees in each angle of the pentagon?

9. An exterior angle of a regular polygon is 9°. How many sides has the polygon?

10. The vertex angle of an isosceles triangle is three times as large as a base angle. How many degrees in each angle?

11. The vertex angle of an isosceles triangle is one-third of a base angle. How many degrees in each angle?

12. How many sides has a regular polygon, each interior angle of which is 172°?

SUGGESTION. — Find the number of degrees in each exterior angle and apply § 134.

13. The sum of two angles of a triangle is 105° and their difference is 25°. How many degrees in each angle of the triangle?

14. The bases of a trapezoid are 17 in. and 9 in. Find the length of the median.

15. The median of a trapezoid is $6\frac{1}{2}$ in. The lower base is 8 in. Find the length of the upper base.

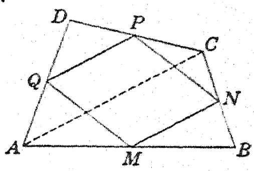
16. In parallelogram $ABCD$, $\angle A = 30°$ and $AD = 32$ in. Find the length of the altitude from D to AB.

GENERAL EXERCISES

1. In $\triangle ABC$, $AB > AC$ and $\angle A = 60°$. Which is the greatest angle of the triangle?

2. AD bisects the base angle of isosceles $\triangle ABC$ with vertex at C, and intersects leg CB in D. Prove $AD > BD$.

3. Prove that the diagonal is greater than a side of a square.



4. The line segments connecting the middle points of the adjacent sides of any quadrilateral form a parallelogram.

SUGGESTION. — Draw diagonal AC. Use § 154.

5. The line segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.

SUGGESTION. — Use Ex. 4.

6. The line segments connecting the middle points of the adjacent sides of a rectangle that is not a square form a rhombus.

7. The quadrilateral which is formed by joining the middle points of the segments into which the diagonals of a given rectangle are divided by their point of intersection is also a rectangle.

8. The line segments connecting the middle points of the adjacent sides of a rhombus form a rectangle. (See § 148, Ex. 1.)

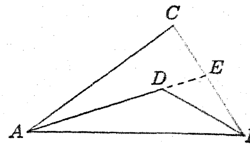
9. The sum of the diagonals of any quadrilateral is greater than the sum of a pair of opposite sides.

10. Any side of a triangle is less than half the perimeter.

11. If two angles of a triangle are unequal, the bisector of the other angle is not perpendicular to the opposite side. (Use indirect proof.)

12. In quadrilateral $ABCD$, AB is the longest side and CD the shortest. Prove $\angle C > \angle A$ and $\angle D > \angle B$.

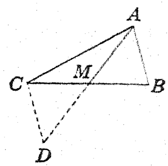
***13.** In any triangle the sum of two sides is greater than the sum of two line segments drawn from a point within the triangle to the extremities of the third side.



SUGGESTION. — Produce AD to meet BC at E. Prove $AC + CB > AE + EB$. Similarly, prove $AE + EB > AD + DB$. Then apply *Ax.* 11.

***14.** The sum of the line segments drawn from any point within a triangle to the three vertices is less than the sum of the three sides. (Use Ex. 13.)

***15.** A median of a triangle is less than half the sum of the two adjacent sides.

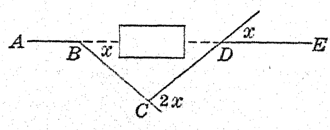


SUGGESTION. — Produce AM to D so that $MD = AM$. $\triangle AMB \cong \triangle DMC$. Why? Then compare $AC + CD$ with AD.

PRACTICAL APPLICATIONS

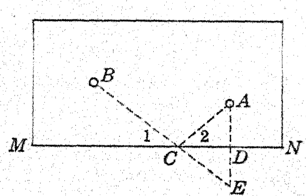
(OPTIONAL)

1. A surveyer extends a straight line AB beyond an obstacle as follows: At B he turns off any convenient angle x, and runs a line to C. At C he turns off an angle equal to $2x$, as in the drawing, and runs a line to D, making $CD = BC$. At D he turns off an angle equal to x, giving the line DE. Prove that AB and DE are in one straight line.



2. When a billiard ball strikes a side of the table, the path along which it rebounds makes an angle with the side of the table equal to that made by the path along which it strikes.

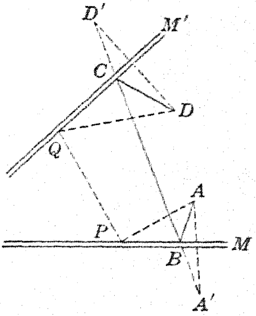
A ball at A is to be driven against the side MN so that when it rebounds it will strike a given ball at B. Show that the following geometric



construction will locate the point C at which the ball must strike the side MN of the table: Draw $AD \perp MN$ and produce it to E so that $DE = AD$, and draw EB intersecting MN at C. (Prove $\angle 1 = \angle 2$.)

3. Prove that the path ACB is the shortest path by which the billiard ball in Ex. 2 could travel from A to B and strike the side MN of the table.

SUGGESTION. — Let the ball strike MN at any other point P, and draw AP, PB, and PE. Prove $AC + CB < AP + PB$.

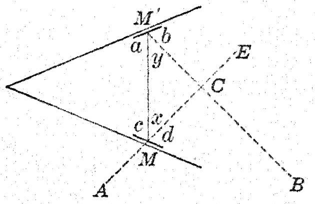


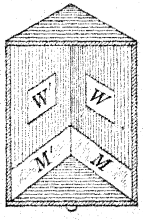
4. Light which strikes a mirror is reflected at an angle with the mirror equal to the angle at which it strikes it.

Two mirrors, M and M', are placed at an angle. Light from a point A travels to the point D along the path $ABCD$. Prove that $ABCD$ is the shortest path that light can travel from A to D and be reflected from both mirrors.

SUGGESTION. — Let $APQD$ be any other path. Prove $ABCD < APQD$. Observe that $ABCD = A'D'$.

5. The optical square is an instrument used in field work such as forestry, for laying off right angles. It consists of a box with triangular top and bottom, and only two side walls, which are set at an angle of 45° with each other. In these walls are cut openings, or

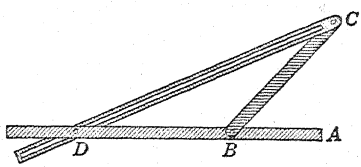




windows, W and W'. Below the windows, mirrors, M and M', are fastened against the walls. The observer, whose eye is at E (see diagram), looks into the box through the open side, and holds it so that an object A can be seen through the window W. At the same time the image of an object B is seen in mirror M, and in line with A. The

principle is that light from B strikes mirror M', is reflected to mirror M, then reflected to the eye at E. By the law of the reflection of light, $\angle a = \angle b$ and $\angle c = \angle d$. Show that $\angle ACB$ is a right angle.

6. An instrument for bisecting an angle consists of three bars AD, BC, CD, pivoted together at B, C, and D. The pivot D works in a groove of the bar CD so that, by lengthening or shortening CD, it is possible to revolve BC about the pivot B and reduce or enlarge $\angle ABC$. BC and BD are equal. Prove that when $\angle ABC$ is adjusted to coincide with any given angle, $\angle BDC$ is equal to one half of that angle.



PRACTICE TESTS

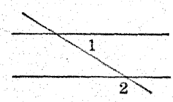
These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT THREE

TEST ONE

Numerical Exercises

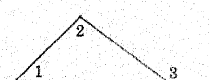
1. If angle 2 is 135° and the lines are parallel, how many degrees in $\angle 1$? § 114.



2. In $\triangle ABC$, $\angle C = 90°$, $\angle A = 30°$, and side $AB = 25$ in. Find side BC. § 160.

3. If one acute angle of a right triangle is five times the other, how many degrees in each? § 125.

4. Angle 1 is 45 degrees and angle 2 is 95 degrees. How many degrees in angle 3? § 128.



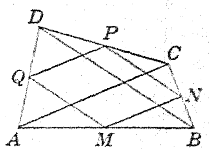
5. One angle of a quadrilateral whose opposite sides are parallel is four times its consecutive angle. Find the number of degrees in each angle. § 115.

6. If the vertex angle of an isosceles triangle is 30 degrees, how large is each base angle? § 123.

7. An interior angle of a regular polygon is 150 degrees. How many sides has the polygon? § 133.

8. In $\triangle ABC$, if AB is bisected at M, and MN is drawn parallel to AC, find the length of MN if AC is 10 inches. §§ 153, 155.

9. An exterior angle of a regular polygon is 60°. How many sides has the polygon? § 134.



10. The quadrilateral $ABCD$ has its sides bisected at M, N, P, and Q. If AC is 16 in. and BD is 26 in., find the perimeter of $MNPQ$. § 155.

11. In Ex. 10, if $\angle CAD$ is 50° and $\angle BAD$ is 70°, how many degrees in $\angle BMN$? § 114.

12. The side of a rhombus is 16 in. and one of its angles is 30°. Find its altitude. § 160.

TEST TWO

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a *rhombus*. § 148.

2. One angle formed by the bisectors of two angles of an equilateral triangle is *double* the third angle of the equilateral triangle. § 123.

3. If the sum of the interior angles is twice the sum of the exterior angles of a regular polygon, the polygon has *eight* sides. §§ 133, 134.

4. If a straight line intersects one of two *parallel* lines, it intersects the other also. § 106.

5. The opposite angles of a parallelogram are *supplementary*. § 141.

6. A line which is perpendicular to one of two *perpendicular* lines is perpendicular to the other. § 109.

7. If two angles have their sides *parallel* each to each, they are either equal or supplementary. § 116.

8. If a quadrilateral is a *square*, its diagonals intersect at right angles. § 148, Ex. 1.

9. If the opposite sides of a quadrilateral are equal, the figure is a *rectangle*. § 149.

10. If a regular polygon has n sides, each interior angle has *180* $(n - 2)$ degrees. § 133.

11. The bisectors of two consecutive angles of a *parallelogram* meet at right angles. §§ 115, 123.

12. In an equiangular triangle, the line joining the mid-points of two sides is equal to *one-sixth* of the perimeter. § 155.

TEST THREE

Multiple-Choice Statements

From the expressions printed in italics select that one which best completes the statement.

1. A hexagon has *five*, $\frac{6(6-3)}{2}$, *six* diagonals. § 130.

2. If two angles have their sides, respectively, perpendicular, they *have their vertices at the same point, are equal or supplementary, are complementary*. § 129.

3. The sum of the exterior angles of a polygon having 12 sides is *one straight angle, the same as the sum of the exterior angles of a hexagon, twelve right angles*. § 134.

4. A trapezoid is a special kind of *quadrilateral, parallelogram, pentagon*. § 140.

5. The consecutive angles of a parallelogram are *supplementary, obtuse, equal* angles. § 115.

6. In a 30°–60° right triangle the side opposite the 30° angle is *half the other side, half the median, half the hypotenuse*. § 160.

7. If one side of a triangle is twice a second side, *the triangle is isosceles, the angle opposite the smaller side is less than the angle opposite the greater side, the angle opposite the greater side is twice as large as the other angle*. § 170.

8. *A hexagon, equilateral triangle, rhombus* is a regular polygon. § 135.

9. A rectangle has the same relation to a square as a parallelogram has to a *rhombus, polygon, quadrilateral*. §§ 137–139.

10. If the median of a triangle is equal to half the side to which it is drawn, the triangle is *a right triangle, a 30°–60° right triangle, an isosceles triangle.* § 123.

11. If two sides of one triangle are equal, respectively, to two sides of another triangle, but the included angle of the first is twice the included angle of the second, then the third side of the first is *twice, greater than, less than* the third side of the second. § 177.

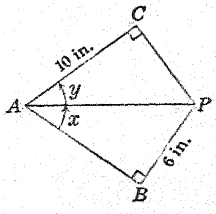
12. If two angles have the initial side of each parallel to the terminal side of the other, the angles are *consecutive angles, equal, supplementary.* §§ 116, 117.

CUMULATIVE TESTS ON THE FIRST THREE UNITS

TEST FOUR

Numerical Exercises

1. Two angles are supplementary. One is 4° less than three times the other. How large is each? § 38.

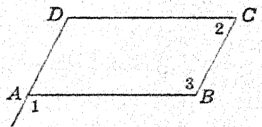


2. In trapezoid $ABCD$, if $AB = 36$ in., and $CD = 20$ in., and $AB \parallel CD$, how long is the median MN? § 157.

3. In $\triangle ABC$, $\angle C = 90°$, $\angle A = 60°$, $AB = 12$ in. How large is AC? § 160.

4. In the figure, $PC \perp AC$, $PB \perp AB$, and $\angle x = \angle y$. How long is PC? § 98.

5. In the same figure, $PC = PB$, $PC \perp AC$, $PB \perp AB$, $\angle x = 25°$. How many degrees in $\angle y$? § 83.

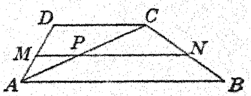


6. In Ex. 5, how long is AB? § 58.

7. $ABCD$ is a parallelogram. If $\angle 1 = 110°$, how many degrees in $\angle 2$? §§ 113, 115.

8. One exterior angle of a regular polygon is 10°. How many sides has the polygon? § 134.

9. The vertex angle of an isosceles triangle is 120°. How large is each base angle? § 123.



10. AB is parallel to DC and M and N are mid-points of the other two sides. If AC is 18 in., how long is PC? §§ 153, 155.

11. Two interior angles of a triangle are 50° and 60°, respectively. How many degrees in the exterior angle at the third vertex of the triangle? § 128.

12. In Ex. 11, how many degrees in the complement of the interior angle at the third vertex? § 123.

TEST FIVE

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. An exterior angle of a triangle is *greater* than either opposite interior angle. § 91.

2. If the four sides of one *quadrilateral* are equal, respectively, to the four sides of another, the *quadrilaterals* are congruent.

3. If a theorem is true, its converse is *always* true. § 74.

4. The *hypothesis* of a theorem is the part that is given; the conclusion is the part to be proved. § 67.

5. If, in $\triangle ABC$, $\angle A = 40°$ and $\angle C = 80°$, the exterior angle at B must be equal to *120°*. §§ 123, 128.

6. If a median of a triangle is perpendicular to the side to which it is drawn, the triangle is *equilateral*. § 64.

7. A diagonal of a *quadrilateral* divides it into two congruent triangles. § 142.

8. If two sides of a triangle are unequal, the angles opposite are unequal, the *greater* angle being opposite the greater side. § 170.

9. A straight line *cannot* be both parallel and perpendicular to another straight line. § 105.

10. If three lines cut off equal segments on two transversals, they *must* be parallel. § 152.

11. If one acute angle of a right triangle is double the other, the *hypotenuse* is double the smaller side. § 160.

12. Of two contradictory propositions, if one is true, the other is *usually* false. § 102, 1.

TEST SIX

Drawing Conclusions

Give a conclusion that can be drawn from the hypothesis stated.

1. In triangle ABC, $AC = BC$. § 69.

2. $\angle a + \angle b = 180°$; $\angle a = \angle c$. § 40.

3. In $\triangle$ ABC and DEF, $AB = DE$, $\angle A = \angle D$, $\angle C = \angle F$. § 127.

4. AB and DC are parallel sides of trapezoid $ABCD$. MN is the median. § 157.

5. In $\triangle ABC$, $AB > AC$. § 170.

6. In $\triangle ABC$, $AB = 6$ in., $\angle A = 55°$, and $\angle B = 60°$. In $\triangle DEF$, $\angle D = 60°$, $\angle F = 55°$, and $DF = 6$ in. § 65.

7. PM is perpendicular to AB and M is the mid-point of AB. § 89*a*.

8. In $\triangle ABC$, $\angle A = \angle B$. § 76.

9. M is the mid-point of hypotenuse AB of right triangle ABC and CM is a median. § 159.

10. $PA = PB$ and $QA = QB$. § 87.

11. Line k passes through point P parallel to line l. Line t intersects k at P. §§ 105, 106.

12. $\angle EBC$ is an exterior angle of $\triangle ABC$. § 128.

TEST SEVEN

Constructions

Make the constructions required, leaving all construction lines.

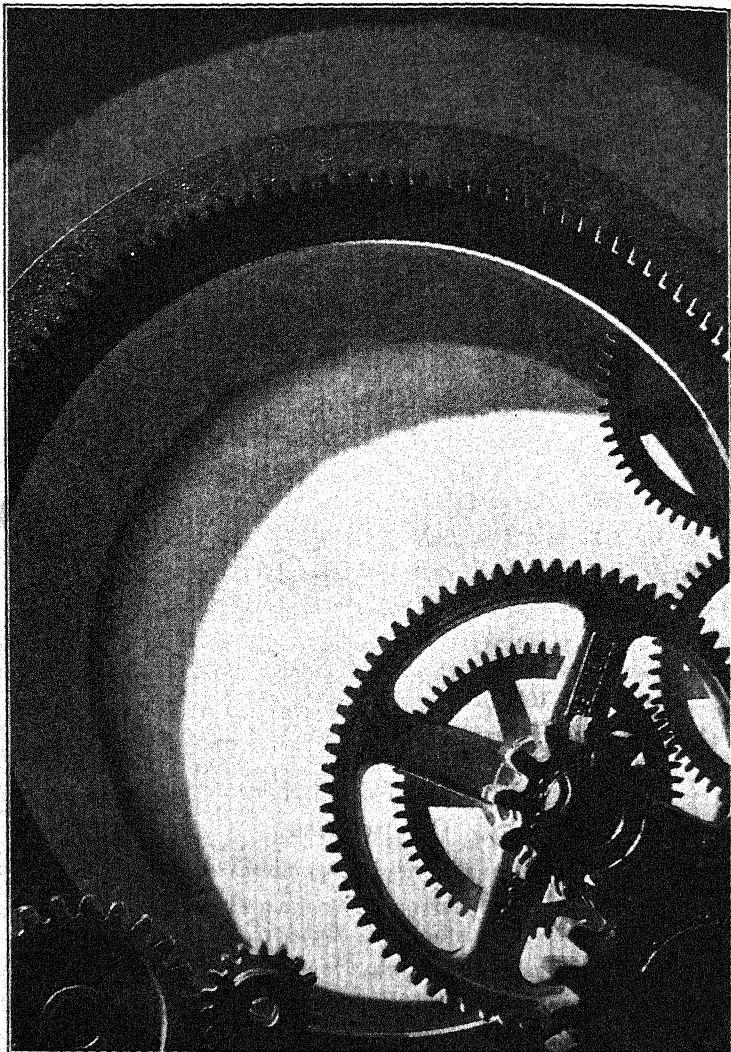
1. Through a given point on a side of a given triangle construct a line parallel to one of the other sides of the triangle.

2. Divide a line $3\frac{1}{2}$ in. long into five equal parts.

3. Construct an angle of 75°.

4. Construct an isosceles triangle given one of the legs and a vertex angle.

5. Construct an isosceles triangle given the base and one of the equal angles.



CIRCLES © *William M. Rittase*

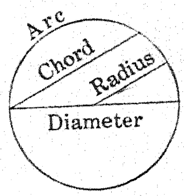
Gears such as these, and many other machine parts are based on the circle.

UNIT FOUR

CIRCLES

Chords, Central Angles, and Arcs; Tangents; Constructions

180. Circles. A **circle** (symbol ⊙) is a closed curve all points of which are equally distant from a point within called the **center.**



A line segment from the center to the circle is called a **radius.**

The **diameter** is a line segment through the center terminated by the circle.

A diameter is equal to two radii.

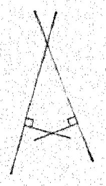
A **chord** is a line segment whose end points are on a circle.

In addition to postulates 5 and 6, which refer to the circle, we shall need the following:

181. Postulate 13. *A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than, the radius.*

182. Postulate 14. *a. A diameter of a circle bisects the circle and the surface inclosed by it,* and conversely

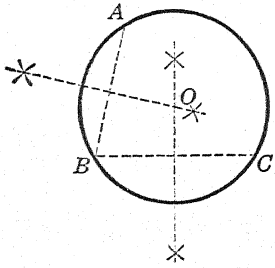
b. If a line bisects a circle, it is a diameter.



183. Postulate 15. *Two lines perpendicular to intersecting lines must intersect.*

Proposition 1. Theorem

184. *Through any three given points not in a straight line one circle, and only one, can be drawn.*



Given: A, B, C, not in a straight line.

To prove: One $\odot$ and only one $\odot$ can be drawn through A, B, and C.

Plan: Draw the perpendicular bisectors of the chords. Prove that they intersect in one and only one point.

Proof:

STATEMENTS	REASONS
1. Draw AB, BC. Construct the perpendicular bisectors of AB and BC, and suppose that they intersect at O.	1. *Post.* 15.
2. A circle can be drawn with O as center and OA as radius.	2. *Post.* 5.
3. $OA = OB = OC$.	3. § 89*a*.
4. The circle will pass through points B and C.	4. *Post.* 13.
5. There can be but one circle through A, B, and C.	5. *Post.* 2.

185. Corollary 1. *A straight line or a circle cannot intersect a circle in more than two points.*

Construction VIII

186. *To construct a circle through three given points not in a straight line.*

Construction and proof: See § 184.

EXERCISES

1. From the measurements of a piece of broken wheel a new wheel is to be cast of the same size. Show how to find the radius of the new wheel.

2. Draw any triangle and construct a circle which will pass through its vertices.

3. Draw a circle by tracing around a coin. Locate its center.

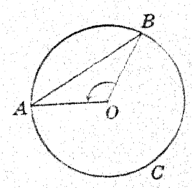
4. Draw two circles that have no point in common; two points in common; one point in common.

---

5. If AB and CD are diameters of the same circle, prove that $\widehat{AC} = \widehat{BD}$.

6. All the vertices of a polygon lie on a circle. Prove that the perpendicular bisectors of the sides pass through the center of the circle.

Central Angles and Their Arcs

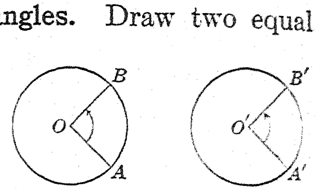


187. A **central angle** is an angle formed by two radii as $\angle BOA$. You see that the vertex is at the center of the circle.

Any part of a circle is called an **arc.** An arc equal to half a circle is called a **semicircle.** In speaking of the arc AB, the arc less than a semicircle, called a **minor arc,** is meant. A **major arc** ($\widehat{ACB}$) is greater than a semicircle.

A **quadrant** is a quarter of a circle.

188. Arcs of equal central angles. Draw two equal circles, O and O'. Construct central angle AOB equal to central angle $A'O'B'$. It can be proved by superposition that $\widehat{A'B'} = \widehat{AB}$. Conversely, if, in the equal circles O and O', $\widehat{A'B'} = \widehat{AB}$, it can be proved that $\angle A'O'B' = \angle AOB$.



We shall assume:

189. POSTULATE 16. *In the same circle or in equal circles*
(*a*) *equal central angles have equal arcs;* and conversely
(*b*) *equal arcs have equal central angles.*

EXERCISES

1. Draw a circle and illustrate a central angle; a minor arc; a major arc.

2. Construct a central angle of 90°; of 45°.

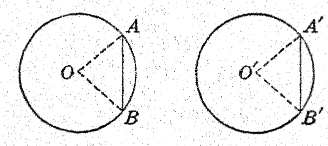
3. Divide a circle into four equal parts.

SUGGESTION. — Draw two perpendicular diameters. Are the central angles equal? Why, then, are the arcs equal?

4. Explain how to divide a circle into eight equal arcs.

5. Is the arc of a 45° central angle half the arc of a 90° central angle? Is it twice the arc of a $22\frac{1}{2}°$ central angle? Prove it.

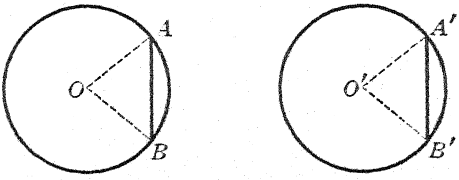
190. Equal arcs and chords. Equal circles O and O' have chords AB and $A'B'$ equal. By drawing OA, OB, $O'A'$, and $O'B'$, see if you can show by congruent triangles and equal central angles that $\widehat{AB} = \widehat{A'B'}$.



Similarly, given that $\widehat{AB} = \widehat{A'B'}$, prove that $AB = A'B'$.

Proposition 2. Theorem

191. *In the same circle, or in equal circles, if two arcs are equal, their chords are equal.*



Given: Equal ⊙ O and O', with $\widehat{AB}=\widehat{A'B'}$.

To prove: $AB = A'B'$.

Plan: Use congruent triangles.

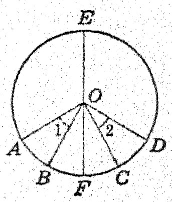
Suggestion. — Think: "Since I know the arcs are equal, I know $\angle O = \angle O'$. (§ 189.) Therefore the △ are congruent (*s.a.s.* = *s.a.s.*), and $AB = A'B'$."

Proof: Write the proof.

192. Concentric circles are circles having the same center.

EXERCISES

1. The radius of a circle is 12 in. How long is a chord whose central angle is 60°?



2. If $\angle 1 = \angle 2$, prove that $\widehat{AC} = \widehat{BD}$.

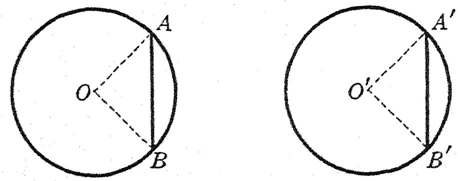
3. If $\angle AOC = \angle BOD$, prove that $\widehat{AB} = \widehat{CD}$.

4. If $AO \perp OC$ and $BO \perp OD$ and diameter FE bisects $\widehat{BC}$, prove $\widehat{AE} = \widehat{DE}$.

5. In the figure, if $\angle 1 = \angle 2$, prove that chord AC is equal to chord BD.

Proposition 3. Theorem

193. *In the same circle, or in equal circles, if two chords are equal, their arcs are equal.*



Given: Equal ⓢ O and O', with $AB = A'B'$.

To prove: $\widehat{AB} = \widehat{A'B'}$.

Plan: Use congruent triangles.

Proof: Write the proof.

EXERCISES

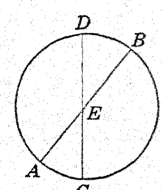
1. Construct three concentric circles with radii $\frac{1}{2}$ in., $\frac{3}{4}$ in., and 1 in.

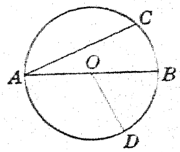
2. Prove that if two chords bisect each other, they must be diameters.

Hint. — Will the center of the circle lie on the perpendicular bisector of AB? Of CD?

3. Using Ex. 2, prove that if chords AB and CD bisect each other, $\widehat{AC} = \widehat{BD}$.

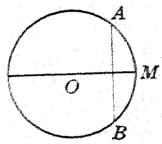
4. Two chords, AB and CD, intersect at point E. If E is the center of the circle, the chords are diameters and $\widehat{AC} = \widehat{DB}$ as in Ex. 3. Imagine that point E moves away from the center toward the mid-point of arc AC, the angle of intersection of the lines remaining the same. In several positions measure the intercepted arcs and see if their sum remains the same.





5. If $\angle DOB = 2\ \angle A$, AB is a diameter and OD a radius, then $\widehat{BD} = \widehat{BC}$.

SUGGESTION. — Draw OC and prove $\angle BOC = \angle DOB$.



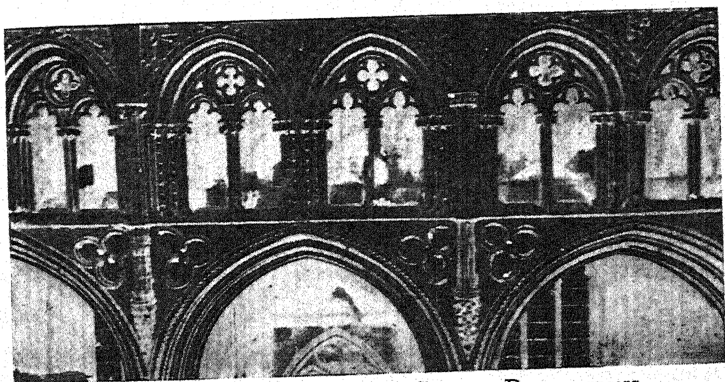
6. A diameter to the mid-point of an arc bisects the chord of the arc and is perpendicular to it.

HINT. — Can you find two points each equidistant from A and B?

7. A diameter perpendicular to a chord bisects the chord and its arcs.

HINT. — Draw OA and OB and prove the triangles congruent.

8. "A diameter which bisects a chord is perpendicular to the chord." Is this theorem *always* true? If it is, prove it; if not, reword it so that it is always true and then prove it.

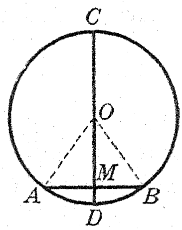


CIRCLES ARE MUCH USED IN CHURCH DECORATIONS

194. A diameter perpendicular to a chord. In Ex. 7 you have proved a theorem about a diameter perpendicular to a chord and in Ex. 8 you have proved the partial converse of this theorem. These form the next two propositions.

Proposition 4. Theorem

195. *A diameter perpendicular to a chord bisects the chord and its arcs.*



Given: Circle O, diameter $CD \perp AB$ at M.

To prove: $AM = MB$, $\overset{\frown}{AD} = \overset{\frown}{DB}$, $\overset{\frown}{AC} = \overset{\frown}{CB}$.

Plan: Use congruent triangles.

Proof:

STATEMENTS	REASONS
1. Draw OA and OB. $OM = OM$.	1. *The same line.*
2. $OA = OB$.	2. *Post.* 6.
3. $\triangle AMO \cong \triangle BMO$.	3. § 83.
4. $AM = MB$. $\angle AOM = \angle MOB$.	4. § 58.
5. $\angle COA = \angle BOC$.	5. § 40.
6. $\overset{\frown}{AD} = \overset{\frown}{DB}$ and $\overset{\frown}{AC} = \overset{\frown}{CB}$.	6. § 189.

Ex. 1. The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs of the chord.

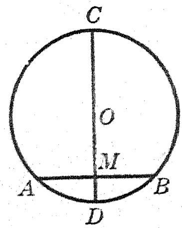
Suggestion. — Where do all points equidistant from the extremities of the chord lie?

Ex. 2. If a line drawn from the center of a circle is perpendicular to a chord, it bisects the chord and its arcs.

Ex. 3. Bisect a given arc of a circle, when the center of the circle is inaccessible.

PROPOSITION 5. THEOREM

196. *A diameter which bisects a chord (not a diameter) is perpendicular to the chord.*



Given: Circle O, with diameter CD intersecting chord AB at its mid-point M.

To prove: $CD \perp AB$.

Analysis: Think: "If CD is perpendicular to AB at the mid-point M, it will be the perpendicular bisector of AB. To prove that CD is the perpendicular bisector of AB I must find two points on CD equidistant from A and B."

Proof: Write the proof.

197. COROLLARY. *The perpendicular bisector of a chord passes through the center of the circle.*

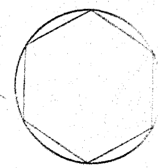
Ex. 1. A line drawn from the center of a circle to the mid-point of a chord is perpendicular to the chord and bisects the arcs of the chord.

Ex. 2. A radius drawn to the mid-point of an arc bisects the chord of the arc and is perpendicular to it.

Ex. 3. The line joining the mid-point of a chord and the mid-point of its arc is perpendicular to the chord and passes through the center of the circle.

Ex. 4. If diameter CD bisects chord AB, the two triangles, CAB and DAB, are isosceles.

198. A circle is said to be **circumscribed** about a polygon when all the vertices of the polygon lie on the circle. The polygon is said to be **inscribed** in the circle.



199. The **line of centers** of two circles is the line passing through their centers.

EXERCISES

1. Using your protractor, how can you divide a circle into five equal arcs? How many degrees in each central angle?

2. Prove that the polygon formed in Ex. 1 by joining the consecutive points on the circle is equilateral.

3. By connecting alternate points in Ex. 1, form a five-pointed star.

4. Divide a circle into eight equal arcs. By joining every third point in order, form an eight-pointed star.

5. Prove that the chord of a $60°$ central angle is equal to the radius.

6. Radius OD is perpendicular to chord AB. Prove that OD bisects angle ADB.

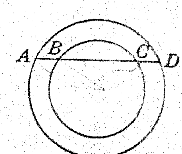
---

7. If a diameter bisects each of two chords, the chords are parallel.

8. Prove that the line connecting the centers of two intersecting circles is the perpendicular bisector of their common chord.

HINT. — Show that two points are equidistant from the ends of the chord.

***9.** Two concentric circles are intersected by a straight line in the successive points A, B, C, and D. Prove that $AB = CD$.

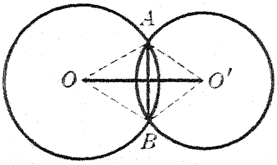


***10.** Two chords AC and AD form $\angle A$, which is bisected by diameter AB. Prove $BC = BD$.

***11.** Prove that the line connecting the centers of two parallel chords passes through the center of the circle.

PROPOSITION 6. THEOREM

200. *If two circles intersect, the line of centers is the perpendicular bisector of their common chord.*

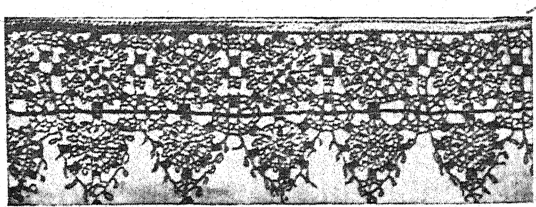


Given: Circles O and O' intersecting in A and B.

To prove: OO' is the perpendicular bisector of AB.

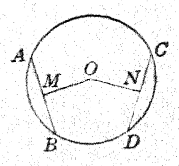
Plan: Points O and O' are each equidistant from A and B. *If two points are each equidistant . . .*

Proof: Write the proof.



CIRCLES USED IN OLD ITALIAN LACE DESIGN

201. The distance from the center of a circle to a chord. What is meant by this (§ 99)? Prove by using congruent triangles that, if $AB = CD$, then $OM = ON$, where $OM \perp AB$ and $ON \perp CD$.

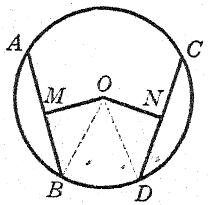


HINT. — Draw OA and OC. Why do OM and ON bisect AB and CD respectively?

Then prove the converse; that is, if $OM \perp AB$ and $ON \perp CD$, and if $OM = ON$, then $AB = CD$.

Proposition 7. Theorem

202. *In the same circle, or in equal circles, chords equidistant from the center are equal.*



Given: Circle O, $OM \perp AB$, $ON \perp CD$, $OM = ON$.

To prove: $AB = CD$.

Plan: Use congruent triangles.

Proof:

STATEMENTS	REASONS
1. Draw OB and OD. $OB = OD$.	1. *Post.* 6.
2. $OM \perp AB$, $ON \perp CD$, $OM = ON$.	2. *Given.*
3. $\triangle OMB \cong \triangle OND$.	3. § 83.
4. $MB = ND$.	4. § 58.
5. But $MB = \frac{1}{2} AB, ND = \frac{1}{2} CD$.	5. § 195.
6. $\therefore AB = CD$.	6. *Ax.* 1 *and* 4.

EXERCISES

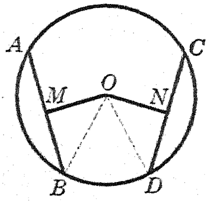
1. In the figure above, if angle OBM is 30° and the radius of the circle is 12 in., how far is chord AB from the center?

2. An isosceles trapezoid is inscribed in a circle of radius 8 in. and a base is a diameter. If the central angle of one of the non-parallel sides is 60°, how long is the other base?

3. The line segment connecting the mid-points of two chords is bisected by the center of the circle. Prove that the chords are equal and parallel.

Proposition 8. Theorem

203. *In the same circle, or in equal circles, equal chords are equidistant from the center.*



Given: Circle O with $AB = CD$, $OM \perp AB$, $ON \perp CD$.

To prove: $OM = ON$.

Plan: Use congruent triangles.

Proof:

STATEMENTS	REASONS
1. Draw OB and OD. $OB = OD$.	1. *Post.* 6.
2. $AB = CD$.	2. *Given.*
3. OM and ON bisect AB and CD, respectively.	3. § 195.
4. $\therefore MB = ND$.	4. *Ax.* 1 *and* 5.
5. $\triangle OMB \cong \triangle OND$.	5. § 83.
6. $\therefore OM = ON$.	6. § 58.

EXERCISES

1. If two equal circles intersect, the common chord bisects the line of centers.

2. In the figure for § 200, if OO' is produced in both directions to meet the circles at P and Q, then P and Q are each equidistant from A and B.

3. The quadrilateral formed by connecting the points of intersection and the centers of two equal intersecting circles is equilateral.

4. Two triangles have as their common base the common chord of two intersecting unequal circles. Their vertices are the centers of the circles. Prove that the triangles are isosceles.

5. Draw a circle and in it draw a large number of equal chords. Why does the figure give the impression of a concentric circle inside the first circle?

6. Divide a circle into four equal arcs. Prove that the chords of these arcs are equidistant from the center.

7. If two equal chords are drawn from the same point on a circle, they make equal angles with the radius drawn to that point.

HINT. — Draw perpendiculars from the center to the chords.

---

8. If the vertices of a polygon lie on a circle and the sides are equidistant from the center, the polygon is equilateral.

9. Two equal chords intersect within a circle. Prove that the radius drawn to the point of intersection bisects the angle between the chords.

HINT. — Draw perpendiculars from the center to the chords.

10. Two chords intersect within a circle, and the radius drawn to the point of intersection bisects the angle between the chords. Prove that the chords are equal.

11. If perpendiculars are drawn to two radii from the mid-point of the arc intercepted by the radii, the perpendiculars are equal.

INEQUALITIES IN CIRCLES

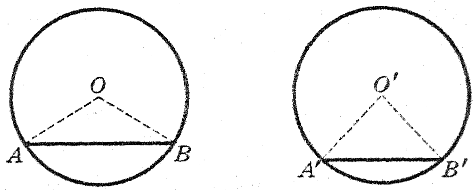
204. Postulates about inequalities in circles.

POSTULATE 17. *In the same circle, or in equal circles,*
a. the greater of two unequal central angles has the greater arc; and conversely

b. of two unequal arcs the greater has the greater central angle.

PROPOSITION 9. THEOREM

205. B. *In the same circle, or in equal circles, if two minor arcs are unequal, the greater arc has the greater chord.*



Given: Equal ⓢ O and O', chords AB and $A'B'$, $\widehat{AB} > \widehat{A'B'}$.

To prove: $AB > A'B'$.

Analysis: If $\widehat{AB} > \widehat{A'B'}$, what can you say about ∡ O and O' (Post. 17)? Since $OA = OB = O'A' = O'B'$, what can you say about AB and $A'B'$ (§ 177)?

Proof:

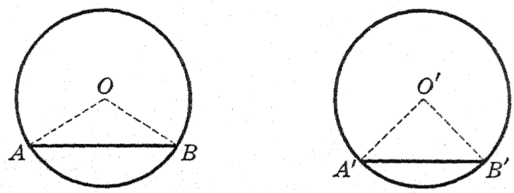
STATEMENTS	REASONS
1. Draw OA, OB, $O'A'$, $O'B'$. They are all equal.	1. *Post.* 6.
2. Since $\widehat{AB} > \widehat{A'B'}$, $\angle O > \angle O'$.	2. *Given and Post.* 17.
3. $\therefore AB > A'B'$.	3. § 177.

The *Sophists*, or "wise men" (about 450 B.C.) made a study of the circle, which had been entirely neglected by Pythagoras and his early followers. They made many attempts to trisect arcs.

The Sophists were the first teachers to receive pay for their work and Hippocrates of Chios (about 430 B.C.) was the first to accept pay for the teaching of mathematics.

PROPOSITION 10. THEOREM

206. B. *In the same circle, or in equal circles, if two chords are unequal, the greater chord has the greater minor arc.*



Given: Equal circles O and O', $AB > A'B'$.

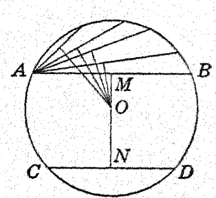
To prove: $\widehat{AB} > \widehat{A'B'}$.

Analysis: Since $AB > A'B'$, we can use § 178 and prove that $\angle O > \angle O'$. Then $\widehat{AB} > \widehat{A'B'}$ by Post. 17.

Proof:

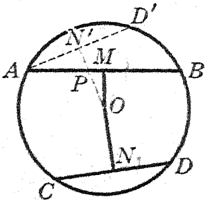
STATEMENTS	REASONS
1. Draw OA, OB, $O'A'$, $O'B'$. They are all equal.	1. *Post.* 6.
2. $AB > A'B'$.	2. *Given.*
3. $\angle O > \angle O'$.	3. § 178.
4. $\widehat{AB} > \widehat{A'B'}$.	4. *Post.* 17.

207. Distances of unequal chords from the center. Does the length of a chord depend on its distance from the center of the circle? If $AB > CD$, how can you place CD to better compare the distances OM and ON? See if you can prove $OM < ON$ by so placing chord CD.



Proposition 11. Theorem

208. B. *In the same circle, or in equal circles, if two chords are unequal, the greater chord is nearer the center.*



Given: Circle O, $AB > CD$, with $OM \perp AB$ and $ON \perp CD$.

To prove: $OM < ON$.

Plan: Take $AD' = CD$ and let $\perp ON'$ intersect AB at P. Then prove $OM < OP$ (§ 173), and that $OP < ON'$ (Ax. 6).

Proof:

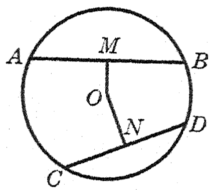
STATEMENTS	REASONS
1. Take $AD' = CD$ and draw $ON' \perp AD'$, intersecting AB at P. $ON' = ON$.	1. *Why is* $AD' < AB$? §§ 203 and 206.
2. $OM < OP$.	2. § 173.
3. $OP < ON'$.	3. *Ax.* 6.
4. Hence $OM < ON'$.	4. *Ax.* 11.
5. $\therefore OM < ON$.	5. *Ax.* 7.

Ex. 1. An equilateral triangle and a square are inscribed in a circle. Prove that the sides of the triangle are nearer the center than the sides of the square.

Ex. 2. Arcs AB, BC, and CD of a circle are equal. From M, the mid-point of arc AB, MN is drawn perpendicular to chord AB at N, and CP is perpendicular to chord BD at P. Prove that $CP > MN$, if each arc is less than a quadrant.

Proposition 12. Theorem

209. B. *In the same circle, or in equal circles, if two chords are unequally distant from the center, the one nearer the center is the greater.*



Given: Circle O, chords AB and CD, $OM \perp AB$ and $ON \perp CD$, $OM < ON$.

To prove: $AB > CD$.

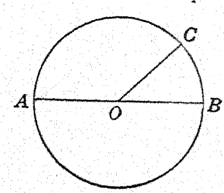
Plan: Use the indirect method of proof.

Proof: Write the proof.

210. A. Continuity. You have seen in §§ 205 to 209 that, as a chord AB moves toward the center of the circle so that the perpendicular gets smaller, the chord AB gets larger and the arc AB also gets larger.

EXERCISES

1. In the same circle or in equal circles, the greater of two unequal major arcs has the smaller chord.



2. In the figure, if AB is a diameter and $\overset{\frown}{BC}$ is less than a quadrant, prove that $\angle COA > \angle BOC$.

3. Triangle ABC is inscribed in a circle. If $AB = 3$ in., $BC = 5$ in., and $CA = 6$ in., prove that $\overset{\frown}{AB} < \overset{\frown}{BC} < \overset{\frown}{CA}$, and that $\angle AOB < \angle BOC < \angle COA$.

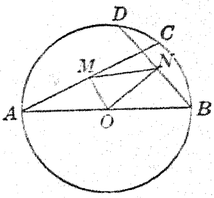
4. A diameter of a circle is greater than any other chord.

5. Prove that the bases of an inscribed trapezoid are unequally distant from the center of the circle.

6. AB is a diameter and chord $BC <$ chord BD. Prove that chord $AD <$ chord AC.

7. In circle O, chord $AB >$ chord BC and chord $BC >$ chord CA. Prove that $\angle AOC < 120°$.

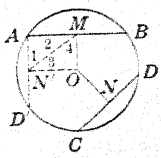
8. If A, B, and C are points on a circle, and P is a point within the circle, and if $\angle APB = \angle BPC$ and $AP = CP$, then $\widehat{AB} = \widehat{BC}$.



9. AB is a diameter and $\widehat{AC} > \widehat{BD}$. If $OM \perp AC$ and $ON \perp BD$, prove $\angle MNO < \angle OMN$.

10. Prove that the shortest chord through a point P within a circle is the chord $AB \perp$ to the diameter through P.

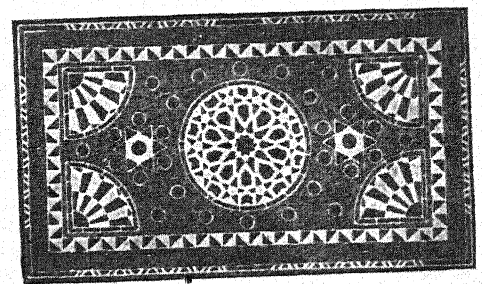
HINT. — Draw any other chord EF through P and show that it is greater than AB by showing that it is nearer the center.



11. Prove the theorems in § 208, § 209 using this figure.

SUGGESTION. — In § 208: if $AD' = CD$, why is $AM > AN'$? Then what can you say about ∡ 1 and 2? About ∡ 3 and 4?

12. Perpendiculars are drawn from a point which is not the mid-point of an arc to the radii that intercept the arc. Prove that the perpendiculars are unequal.

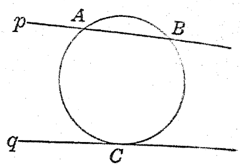


WOOD PANEL INLAID WITH IVORY AND EBONY

SECANTS AND TANGENTS

211. A **secant** is a straight line intersecting a circle in two points.

Thus, secant p intersects the circle in points A and B.



212. A **tangent** is a straight line touching a circle in one, and only one point.

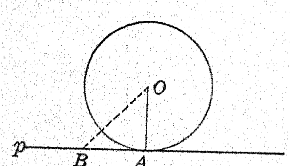
Thus, q is a tangent, and C is called the **point of tangency** or **point of contact.**

Ex. 1. Draw a circle with center O and radius OA. Through A draw a line p making an angle of about 30° with OA. In how many points does p intersect the circle?

Ex. 2. Let the line p in Ex. 1 revolve about the point A so as to make the angle between OA and p larger. How large do you think the angle is when p becomes tangent to the circle?

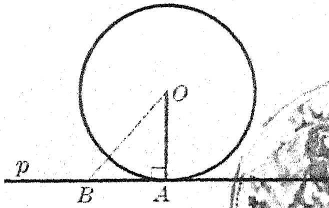
Ex. 3. Secant p intersects a circle in points A and B. Let p revolve about A so that B approaches A along the curve. When B coincides with A, p is said to be tangent to the circle at A. This is the definition of tangent used in higher mathematics.

213. Looking ahead. In order to prove that line p is tangent to the circle at point A, what is it necessary to prove (§ 212)? How can you show that all points of p except A lie outside the circle O? (Post. 13.) By taking any point in p except A, such as point B, and drawing OB, see if you can show that a line p, which is perpendicular to radius OA at A, is tangent to the circle. (See § 173.)



Proposition 13. Theorem

214. *A line perpendicular to a radius at its outer extremity is tangent to the circle.*



Given: Circle O, with $p \perp$ radius OA at A.

To prove: p is tangent to $\odot O$.

Plan: Show that any other line from O to p is greater than OA; and hence that every point of p except A lies outside the circle.

Proof:

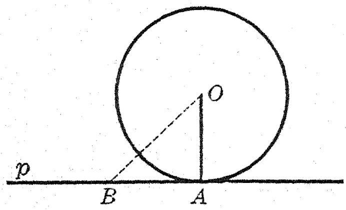
STATEMENTS	REASONS
1. Let B be *any* point on p except A. Draw OB. $OA \perp p$.	1. *Given.*
2. $OB > OA$.	2. § 173.
3. $\therefore B$ lies outside circle O.	3. *Post.* 13.
4. Since B, *any* point on p except A, lies outside circle O, p is tangent to circle O at A.	4. § 212.

Ex. 1. Perpendiculars to a diameter at its extremities are tangent to the circle.

Ex. 2. If the vertices of a regular hexagon are on a circle, a concentric circle can be drawn to which all sides of the hexagon are tangent.

PROPOSITION 14. THEOREM

215. *The tangent to a circle at a given point is perpendicular to the radius drawn to that point.*



Given: Circle O with radius OA, and line p tangent to $\odot$ O at A.

To prove: $OA \perp p$.

Plan: Mentally: "If I can show that OA is the shortest line from O to p, then OA will be the perpendicular." Recall § 174.

Proof:

STATEMENTS	REASONS
1. Take B any point on p except A and draw OB. B lies outside the $\odot$.	1. § 212.
2. $OA < OB$.	2. *Post.* 13.
3. $\therefore$ since $OA <$ any other line drawn from O to p, it is $\perp$ p.	3. § 174.

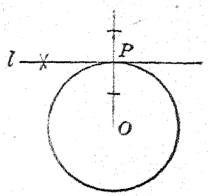
216. COROLLARY 1. *A line perpendicular to a tangent at the point of contact passes through the center of the circle.*

217. COROLLARY 2. *A line from the center of a circle, perpendicular to a tangent, passes through the point of contact.*

SUGGESTION. — Draw a $\perp$ to p at A. Use § 216.

Construction IX

218. *To construct a tangent to a circle through a given point on the circle.*



Given: Point P on circle O.

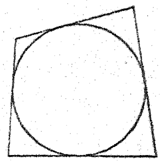
Required: To construct a tangent to $\odot O$ at P.

Construction:

1. Draw OP.
2. Construct $l \perp OP$ at P.
3. l is the required tangent.

Proof: Use § 214.

219. Inscribed circles. A circle is said to be **inscribed** in a polygon when all the sides of the polygon are tangent to the circle. The polygon is said to be **circumscribed** about the circle.

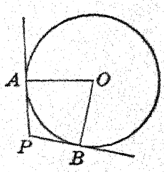


EXERCISES

1. Draw a circle and its diameter AB. At A and B construct tangents to the circle. Prove that the tangents are parallel.

2. Construct a tangent to a circle at the mid-point C of arc AB. Prove that the tangent is parallel to chord AB.

3. Draw any chord of a circle and construct two tangents, each parallel to chord AB.



4. Draw a circle and two radii OA and OB. Construct tangents to the circle at A and B, intersecting at P. How large is the angle at P if $\angle O = 60°$? $90°$? $120°$?

5. In the figure for Ex. 4 prove that $PA = PB$.

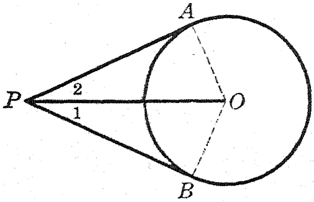
Suggestion. — Draw PO.

---

6. The diameter of a circle bisects all chords which are parallel to the tangent at the extremity of the diameter.

Proposition 15. Theorem

220. *Two tangents to a circle from an outside point are equal and make equal angles with the line joining that point to the center.*



Given: Circle O, with point P outside the circle, PA and PB tangents at A and B, respectively, and line PO connecting points P and O.

To prove: $PA = PB$, and $\angle 1 = \angle 2$.

Plan: Use congruent triangles.

Proof: Left for you to write in full.

Hint. — Why is $OA \perp PA$?

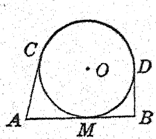
EXERCISES

1. Using the figure in § 220, if PA and PB are tangents to the circle at points A and B, respectively, prove that PO is the perpendicular bisector of chord AB.

2. In § 220 prove that chord AB makes equal angles with the tangents PA and PB.

3. If $\angle BPA$ is 60° and AO is 12 in., how long is PO?

4. If PA is 8 in. and PO is 16 in., how large is $\angle AOB$?



5. AB is tangent to the circle with center O at M. AC and BD are tangent at C and D, respectively. Prove that $AB = AC + BD$.

6. Draw a circle O and any radius OP. On OP produced take point O'. With O' as center and $O'P$ as radius draw a circle. At P construct line l perpendicular to OO'. How is line l related to circles O and O'? Prove it.

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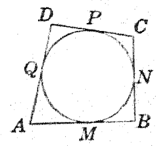
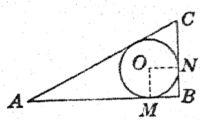
7. If an isosceles triangle is circumscribed about a circle, the base is bisected at the point of contact.

8. A parallelogram circumscribed about a circle is either a rhombus or a square.



9. If a circle is inscribed in an equilateral triangle, the three sides are bisected at the points of contact.

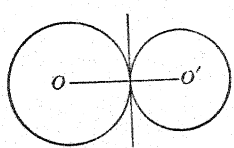
10. In a circumscribed quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.



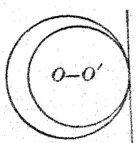
*11. If a circle is inscribed in a right triangle, the sum of the legs equals the sum of the hypotenuse and the diameter of the circle.

*12. The perimeter of any circumscribed trapezoid is equal to four times the line segment which joins the middle points of the two non-parallel sides.

SUGGESTION. — See § 157.



I

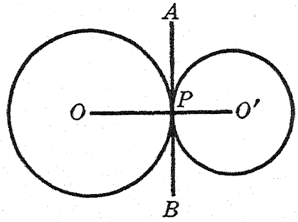


II

221. Tangent circles. Two circles are said to be tangent if they are both tangent to the same line at the same point. If the line of centers is intersected by the common tangent, the circles are said to be **tangent externally** (Fig. I). They are **tangent internally** if the common tangent does not intersect the line of centers (Fig. II).

PROPOSITION 16. THEOREM

222. *If two circles are tangent, the line of centers passes through the point of contact.*



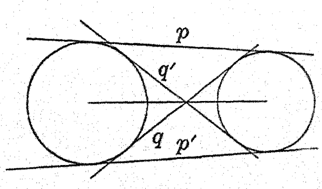
Given: Circles O and O' tangent to AB at P.

To prove: OO' passes through P.

Plan: At P construct a perpendicular to AB and extend it both ways. Why will it pass through O? Through O' (§ 216)?

Proof: For you to write out.

223. Internal and external tangents. A straight line tangent to each of two circles is called a **common tangent.** The tangent is a **common internal tangent** if it intersects the line segment joining their centers; if it does not, it is a **common external tangent.** The length of the common tangent is measured between the two points of contact. Thus, in the figure, p and p' are common external tangents and q and q' are common internal tangents.



224. The **perimeter** of a polygon is the sum of the lengths of its sides.

EXERCISES

1. Draw two circles O and O' with radii r and r', respectively, so placed that $OO' > r + r'$. Draw the common internal and external tangents. How many of each are there?

2. Draw two circles as in Ex. 1 with $OO' = r + r'$. How many common internal tangents are there? How many common external tangents?

3. If two circles are tangent internally, what is the relation between the length of the segment connecting their centers and their radii?

4. In Ex. 3 how many common internal tangents are possible? How many common external tangents?

5. If one circle lies within another circle, and they are not tangent, is $OO' < r - r'$? Do the circles have any common tangents?

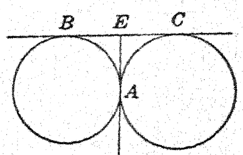
6. Are the common internal tangents of two unequal circles equal? Prove it. (Use § 220.)

7. Prove that the common external tangents of two unequal circles are equal.

HINT. — Produce the tangents until they meet, then use § 220.

8. Draw two circles having only three common tangents.

9. In the figure of § 222, prove that tangents drawn to circles O and O' from any point A in AB are equal.



10. Two circles are tangent externally at A, and also have a common tangent touching them at B and C, respectively. Prove that the common tangent at A bisects BC.

11. In the figure of Ex. 10, prove that $\angle CAB$ is a right angle.

*12. If two circles are tangent externally at A and have a common tangent touching them at B and C, respectively, a circle with diameter BC will pass through A.

*13. Write out a proof for the theorem in Ex. 9 when the two circles are tangent internally.

MISCELLANEOUS EXERCISES

1. Three circles with radii 4 in., 5 in., and 6 in. are tangent externally, each one to the other two. Find the perimeter of the triangle formed by joining their centers.

2. If two circles are tangent to each other, the distance between their centers equals the sum or difference of their radii.

3. If two equal circles are tangent externally at A, and a line is drawn through A intersecting the circles again at B and C, respectively, the chords AB and AC are equal.

SUGGESTION. — Draw the line of centers and radii to B and C.

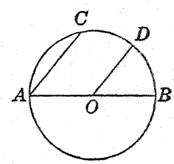
4. A common internal tangent to two equal circles bisects the line of centers.

5. Construct a circle concentric to a given circle and tangent to a chord of the given circle.

6. Given a line l and a point P on l. Construct two circles tangent to l at P, and having given radii r and r'.

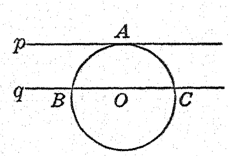
7. Two chords perpendicular to a third chord at its extremities are equal.

8. In circle O, AB is a diameter and $AC \parallel OD$. Prove $\overset{\frown}{CD} = \overset{\frown}{BD}$.

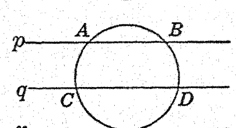


9. Prove that the straight line connecting the mid-points of the major and minor arcs of a chord is perpendicular to and bisects the chord.

225. Parallel lines and the arcs they intercept. If $p \parallel q$, and p is tangent to the circle at A, do you think you can prove $\overset{\frown}{AB} = \overset{\frown}{AC}$? What auxiliary line can you draw which will show that $\overset{\frown}{CA} = \overset{\frown}{AB}$ (§ 195)? If OA is drawn, will it be perpendicular to p and q (§§ 215 and 109)?

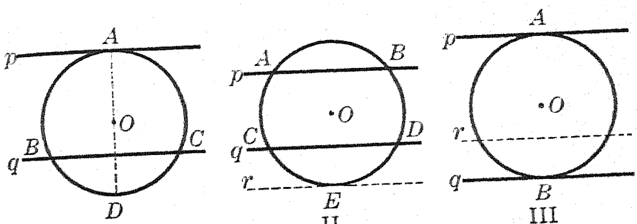


If you can prove that a tangent and a secant intercept equal arcs, see if you can prove that the arcs AC and BD made by parallel secants are equal.



PROPOSITION 17. THEOREM

226. *Two parallel lines intercept equal arcs on a circle.*



I II III

CASE I. *When one parallel is a tangent and one a secant.*

Given: p tangent to $\odot\, O$ at A, q intersecting $\odot\, O$ at B and C, $p \parallel q$.

To prove: $\overset{\frown}{AB} = \overset{\frown}{AC}$.

Plan: Think: "If I draw diameter AD, it will be $\perp\, p$ (§ 215), and hence $\perp\, q$ (§ 109). But a diameter $\perp$ to a chord . . ." (§ 195).

Proof:

STATEMENTS	REASONS
1. Draw diameter AD. $AD \perp p$.	1. § 215.
2. $AD \perp q$.	2. § 109.
3. $\therefore \overset{\frown}{AB} = \overset{\frown}{AC}$.	3. § 195.

CASE II. *When both parallels are secants.* (Fig. II.)

SUGGESTION. — Suppose r drawn $\parallel p$ and tangent to $\odot\, O$ at E.

1. Why is $r \parallel q$?
2. Why is $\overset{\frown}{AE} = \overset{\frown}{BE}$?
3. Why is $\overset{\frown}{CE} = \overset{\frown}{DE}$?
4. Then why is $\overset{\frown}{AC} = \overset{\frown}{BD}$?

CASE III. *When both parallels are tangents.* (Fig. III.)

SUGGESTION. — Construct a secant $r \parallel p$. Then apply Case I.

EXERCISES

1. Using Fig. III, prove that a diameter bisects the circle.

2. In Fig. II, prove that $\widehat{AE} = \widehat{BE}$.

3. In Fig. II, prove that $\widehat{ABD} = \widehat{BAC}$.

4. An inscribed trapezoid is isosceles.

5. The diagonals of an inscribed trapezoid are equal.

6. If two circles are concentric, chords of the larger which are tangent to the smaller are equal.

---

7. If a tangent and a secant intercept equal arcs, they are parallel.

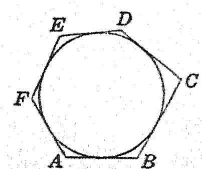
HINT. — Use the indirect method. In Fig. I, § 226, if q is not $\parallel p$, through B suppose a line drawn $\parallel p$.

8. If two secants which do not intersect within a circle intercept equal arcs, they are parallel.

HINT. — See Ex. 7 and use Fig. II.

9. The opposite sides of an inscribed equilateral hexagon are parallel.

HINT. — Use the result obtained in Ex. 8.



10. In any circumscribed hexagon, the sum of one set of alternate sides equals the sum of the other set; that is,

$$AB + CD + EF = BC + DE + FA.$$

11. In any circumscribed octagon, the sum of one set of alternate sides equals the sum of the other set.

12. If from an external point P two tangents are drawn to a circle with center O, the points of contact being A and B, then $\angle BPA = 2\angle OBA$.

*13. The sides of a triangle circumscribed about a circle are 5 in., 6 in., and 8 in. Find the lengths of the segments into which the sides are divided at the points of contact with the circle.

SUGGESTION. — Form three equations and solve for the three unknown quantities. See *Review of Algebra*, page 447.

Constructions

227. You have had the following constructions so far in this course:

I. To construct the perpendicular bisector of a segment (§ 92).

II. To construct an angle equal to a given angle (§ 93).

III. To construct a perpendicular to a line at a given point on the line (§ 94).

IV. To construct a perpendicular to a given line from a given outside point (§ 95).

V. To construct the bisector of an angle (§ 96).

VI. To construct a parallel to a given line through a given point (§ 122).

VII. To divide a given line into any number of equal parts (§ 162).

VIII. To construct a circle through three given points not in a straight line (§ 186).

IX. To construct a tangent to a circle through a given point on the circle (§ 218).

Review each of these constructions now.

228. Constructing triangles. You already know how to construct triangles in the following simple cases.

Construction X. *Construct a triangle when three sides are given.* (See § 17.)

Construction XI. *Construct a triangle when two sides and the included angle are given.* (See § 52.)

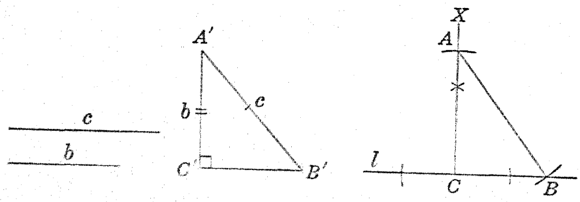
Construction XII. *Construct a triangle when two angles and the included side are given.* (See § 53.)

Review these constructions now.

CONSTRUCTING RIGHT TRIANGLES

Construction XIII

229. *To construct right $\triangle ABC$, given hypotenuse c and side b.*



Plan: $\triangle A'B'C'$ is a freehand drawing of the required construction. A study of it indicates the method of making the construction.

Construction:

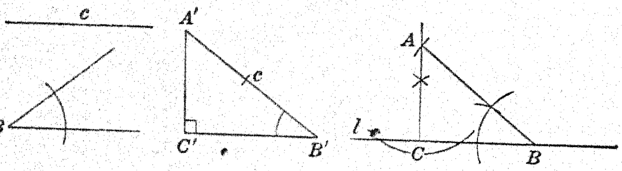
1. At any point C on line l construct $CX \perp l$.
2. From C on CX take $CA = b$.
3. With A as a center and a radius c cut l in B.
4. Draw AB.
5. ABC is the required triangle.

Proof: The proof is obvious: AC was constructed $\perp BC$, $AB = c$ and $AC = b$.

Discussion: The problem is impossible if $c < b$ or if $c = b$.

Construction XIV

230. *To construct right $\triangle ABC$, given hypotenuse c and acute $\angle B$.*



Plan: $\triangle A'B'C'$ represents our freehand drawing. It is evident that we can construct $\angle B$, take c on one of its sides, and drop a $\perp$ from the extremity of c to the other side.

Construction:

1. At any point B on line l copy $\angle B$.
2. On one side of the angle take $BA = c$.
3. From A draw $AC \perp l$.
4. ABC is the required triangle.

Proof: AC was constructed $\perp$ BC. We have used the given parts and hence ABC is the required triangle.

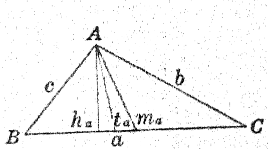
Discussion: There is no solution unless $\angle B < 90°$.

231. From constructions X–XIV you see that, in general, a triangle can be constructed if three parts are given, provided at least one of the parts is a side.

232. How to make a construction.

1. *Draw the figure freehand and mark the given parts.*
2. *Try to discover a part of the figure which you can begin to construct.*
3. *Actually make the construction and complete the figure, using only an unmarked straightedge and compasses.*
4. *Prove that your figure satisfies the given requirements.*
5. *Discuss the limitations the solutions may have, and the possibility of more than one solution.*

233. Symbols. For convenience we shall designate the angles of a triangle by the capital letters A, B, and C; the sides opposite these angles by the corresponding small letters a, b, and c; medians to the sides by m_a, m_b, and m_c; altitudes by h_a, h_b, and h_c; and the bisectors of the angles by t_a, t_b, and t_c.



EXERCISES

1. Construct $\triangle ABC$ if $\angle A = 45°$, $b = 3$ in., and $c = 2$ in.

HINT. — Bisect a right angle to get 45°.

2. Construct $\triangle ABC$ if $\angle A = 22\frac{1}{2}°$, $\angle B = 60°$, and $c = 2$ in.

HINT. — Construct an equilateral triangle to get an angle of 60°.

3. Construct $\triangle ABC$ if $\angle C = 90°$, $\angle A = 30°$, and $a = 1\frac{1}{2}$ in. (How can you determine $\angle B$?)

4. Construct a right $\triangle ABC$ having given a leg and the opposite acute angle.

5. Construct a $\triangle ABC$ having given a side, an angle opposite that side, and another angle.

6. Construct an isosceles triangle with base $AB = 2$ in. and $\angle A = 30°$.

7. Construct an isosceles triangle with base $c = 2$ in. and the altitude on the base $(h_c) = 3$ in.

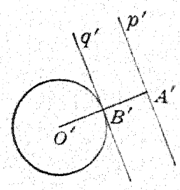
8. Construct an isosceles triangle with base $c = 1\frac{1}{2}$ in. and vertex $\angle C = 45°$.

HINT. — If vertex $\angle C$ is given, what is the sum of the base angles? How can you find one of them?

9. Construct an isosceles triangle with leg $a = 1\frac{1}{2}$ in. and altitude $h_a = 1$ in.

MORE DIFFICULT CONSTRUCTIONS

1. To a given circle construct a tangent parallel to a given line.

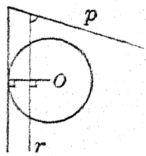


SUGGESTION. — The figure is a sketch of the completed construction. If p' is the given line, q' the required tangent, B' the point of contact, and $p' \parallel q'$, will $O'B'$ be $\perp$ to p'? How can B' be located?

2. To a given circle construct a tangent perpendicular to a given line.

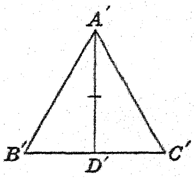
HINT. — Suppose the construction completed, and draw a sketch of it. If the given line is a secant, will the construction be possible?

3. To a given circle construct a tangent making a given angle with a given line.



HINT. — If any line r makes an angle equal to the given angle with the given line p, how should the tangent be constructed?

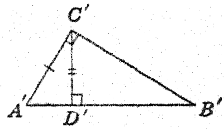
4. Construct an equilateral triangle having a given altitude.



SUGGESTION. — $A'B'C'$ is the freehand drawing. How large is $\angle A'D'B'$? $\angle B'$? Can you construct $\triangle A'B'D'$?

5. Construct an isosceles triangle having a given vertex angle and a given altitude from that vertex.

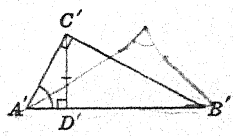
6. Construct a right triangle, having given one leg and the altitude to the hypotenuse.



SUGGESTION. — If $A'B'C'$ represents a freehand sketch of the completed construction, which of the triangles can you construct? (See § 229.)

Having constructed $\triangle ACD$, produce AC through C and construct $CB \perp AC$, meeting AD produced in B. Prove your construction correct.

7. Construct a right triangle, having given an acute angle and the altitude to the hypotenuse.

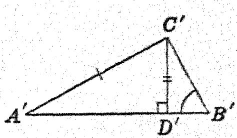


HINT. — Can you construct $\triangle ACD$?

8. Construct a triangle, having given a side, the median to that side, and the altitude to that side.

OPTIONAL CONSTRUCTION EXERCISES

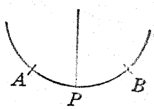
1. Construct a triangle, having given side b, $\angle B$, and the altitude to side c.



SUGGESTION. — Having constructed $\triangle ACD$, how can you find $\angle DCB$ if you know $\angle BDC$ and $\angle B$ (§ 123)?

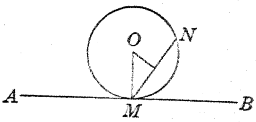
2. Construct a triangle, having given two sides and the altitude to one of the given sides.

3. Draw a tangent to a given circle at a given point when the center is inaccessible.



HINT. — Take $PA = PB$. Will a line from $P \perp AB$ be a diameter if produced?

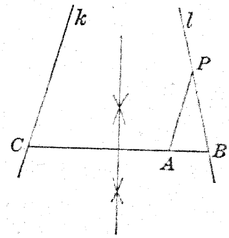
4. Construct a circle which shall be tangent to a given line at a given point and pass through a given external point.



ANALYSIS. — If the required circle is tangent to AB at M, radius $OM \perp AB$. If it passes through M and N, MN is a chord, and hence O is on the perpendicular bisector of MN. Hence O is the intersection of the perpendicular to AB at M and the perpendicular bisector of MN.

Make the construction, and write the complete construction and proof.

5. Construct the bisector of a given angle when the vertex is inaccessible.



HINT. — Let k and l be the given lines. Through any point P of l construct a line $\parallel k$. If PA be taken equal to PB, show how BA produced to C will be the base of an isosceles triangle whose legs are k and l. Will the perpendicular bisector of BC pass through the vertex of this isosceles triangle?

6. Construct a circle with given radius which shall pass through two given points.

Construct triangles, given:

7. a, c, h_b **8.** a, h_a, B **9.** a, b, m_a

10. A, h_a, t_a **11.** a, B, t_b ***12.** a, b, A

234. Summary of the Work of Unit Four. In Unit Four you have learned:

I. ***You can prove two segments are equal by proving that, in the same or in equal circles:***

1. *They are chords having equal central angles, or equal arcs.*

2. *They are chords equidistant from the center of the circle.*

II. ***You can prove two angles are equal by proving that, in the same or in equal circles:***

1. *They are central angles having equal arcs or chords.*

III. ***You can prove two arcs are equal by proving that, in the same or in equal circles:***

1. *They are arcs having equal chords or central angles.*

IV. ***You can prove two segments or two angles are unequal by proving that, in the same or in equal circles:***

1. *They are chords having unequal central angles or unequal arcs.*
2. *They are central angles having unequal chords or arcs.*

V. ***You can prove two arcs are unequal by proving that, in the same or in equal circles:***

1. *They have unequal central angles or unequal chords.*

REVIEW OF UNIT FOUR

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the tests. The references given are those most closely related to the exercise.

1. Define the following terms:

a. minor arc. § 187.
b. major arc. § 187.
c. chord. § 180.
d. perimeter. § 224.
e. semicircle. § 187.
f. central angle. § 187.
g. quadrant. § 187.
h. concentric circles. § 192.
i. circumscribed circle. § 198.
j. inscribed polygon. § 198.
k. tangent. § 212.
l. secant. § 211.
m. common external and internal tangents. § 223.
n. tangent circles. § 221.

Complete the following:

2. A circle may be drawn through any —— points . . . . . . § 184.

3. Two circles may intersect in —— points at most. § 185.

4. A line may intersect a circle in —— points at most. § 185.

5. If central angles in a circle are equal, they have equal —— and ——. §§ 189, 191.

6. Unequal chords in a circle have unequal —— and ——. §§ 204, 206.

7. If a diameter bisects a chord . . . . . . § 196.

8. If a diameter is perpendicular to a chord . . . . § 195.

9. Chords in a circle equidistant . . . . § 202.

10. A line perpendicular to a radius at its outer extremity . . . . . § 214.

11. A line perpendicular to a tangent at the point of contact . . . . . . § 216.

12. A line from the center of a circle, perpendicular to a tangent . . . . . § 217.

13. Two tangents to a circle from an outside point . . . . . § 220.

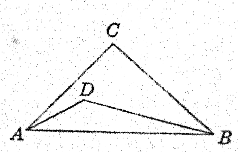
14. State the postulates applying to a circle. §§ 18, 181, 182, 189, 204.

15. Tell how to construct a triangle when you are given (1) two sides and the included angle; (2) two angles and the included side; (3) three sides. § 228.

16. Tell how to construct a right triangle when you are given (1) the hypotenuse and a side; (2) the hypotenuse and an acute angle. §§ 229, 230.

17. Tell how to construct angles of 90°, 135°, $112\frac{1}{2}$°, 45°, 60°, 30°, 15°.

GENERAL EXERCISES



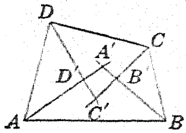
1. D is a point within $\triangle ABC$, such that $BD = BC$. Prove $AC > AD$.

2. If a line is drawn bisecting an angle of any parallelogram, it cuts off an isosceles triangle from the parallelogram.

3. If lines are drawn bisecting two consecutive angles of a parallelogram, the segments of the bisectors included by the sides of the parallelogram bisect each other.

4. The triangle formed by joining the middle points of the sides of an isosceles triangle is isosceles.

5. If the vertices of a quadrilateral lie on a circle, the perpendicular bisectors of all the sides intersect at one point.



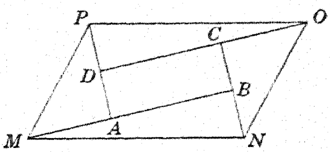
6. If the bisectors of the angles of quadrilateral $ABCD$ form a quadrilateral $A'B'C'D'$, the opposite angles of the quadrilateral $A'B'C'D'$ are supplementary.

7. Through a given point within a circle draw a chord which shall be bisected at the point.

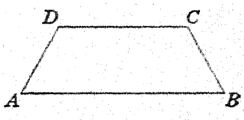
8. If a transversal cuts two parallel lines, the bisectors of the four interior angles form a rectangle.

9. The arcs intercepted between a diameter and a parallel chord are equal.

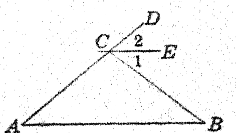
10. The quadrilateral formed by the bisectors of the angles of a parallelogram is a rectangle.



11. The sides of $\square\ ABCD$ are produced in succession to points M, N, O, and P, so that $DP = BN$ and $AM = CO$. Prove that $MNOP$ is a parallelogram.



12. In quadrilateral $ABCD$, if $AD = BC$ and $\angle D = \angle C$, then $DC \parallel AB$.



13. If the bisector of an exterior angle at a vertex of a triangle is parallel to the opposite side, the triangle is isosceles.

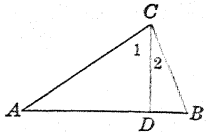
Given: CE bisects $\angle BCD$, an exterior angle of triangle ABC, and $CE \parallel AB$.

To prove: $\triangle ABC$ is isosceles.

*14. If lines are drawn bisecting the angles of a parallelogram, the diagonals of the rectangle formed by the bisectors are parallel to the sides of the parallelogram.

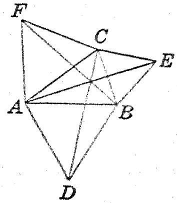
*15. If the exterior angles at A and B of $\triangle ABC$ are bisected by AD and BD, respectively, then $\angle ADB = 90° - \frac{1}{2}\angle C$.

*16. The perpendicular from a vertex of any triangle to the opposite side divides the angle at the vertex into two angles whose difference equals the difference between the other two angles of the triangle. ($\angle 1 - \angle 2 = \angle B - \angle A$.)



*17. The line-segments which join the middle points of the adjacent sides of an isosceles trapezoid form a rhombus or a square.

*18. If $\triangle ABD$, $\triangle BCE$, and $\triangle ACF$ are equilateral triangles drawn on the sides of $\triangle ABC$, AE, BF, and CD are equal.

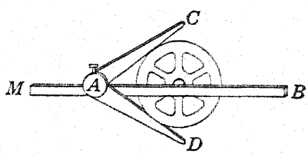


SUGGESTION. — Prove $\triangle BCF \cong \triangle ACE$, etc.

PRACTICAL APPLICATIONS

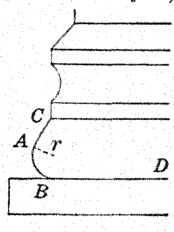
(OPTIONAL)

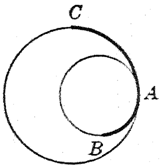
1. The instrument called a *center square* is used for locating the centers of circular objects. It consists of a steel blade M, upon which slides an attachment with two prongs, AC and AD. The edge AB of the blade M bisects the angle between the prongs AC and AD. When the center of a circular object is to be found, the instrument is placed so that the prongs AC and AD are tangent to it. Prove that when this is done, AB passes over the center of the object.

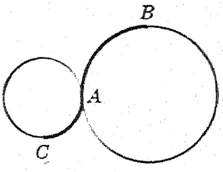


2. The prongs AC and AD of the center square in Ex. 1 are equal. Prove that if the circular object is so large that when the instrument is applied to it the ends of the prongs, C and D, rest against the object, AC and AD becoming secants, then AB passes over the center of the object.

3. In this base of a *column*, straight segments AC and BD are given, and it is required to connect them by a circular arc with a given radius r. Show how to make a drawing on a large scale of the whole base of a column like this one.

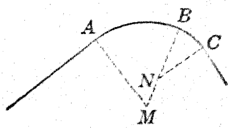






4. Arcs AB and AC of two circles, respectively, which are tangent to each other at A, form a *compound curve*. The point of contact A is the transition point from one circle to the other.

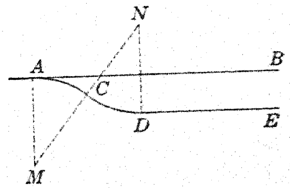
Compound curves are used in the construction of railroads where the winding track must be made to conform to the physical features of the land. They also secure *easement*, preventing a train from lurching when it comes to a curve. The curve ABC of a railroad track is composed of two arcs, $\widehat{AB}$ and $\widehat{BC}$. The center N of $\widehat{BC}$ is on the radius MB of $\widehat{AB}$. Prove that ABC is a *compound* curve, or that $\widehat{AB}$ and $\widehat{BC}$ are arcs of tangent circles.



SUGGESTION. — Draw a line through B perpendicular to BM.

5. AB is a straight railroad track. The track of a switch $ACDE$ is laid out as follows: Arc AC is constructed with center M. Then MC is produced to N, making $CN = MC$, and $\widehat{CD}$ drawn with center N.

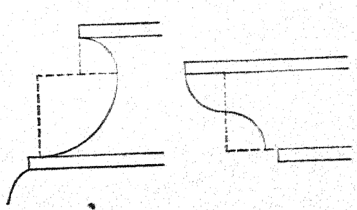
Prove that ACD is a *compound* curve, or that $\widehat{AC}$ and $\widehat{CD}$ are arcs of tangent circles.



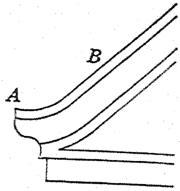
6. Compound curves are used in many kinds of molding, as well as in other architectural designs.

Explain the construction of the curves in the adjoining drawings of moldings.

Make a large drawing of such moldings. Draw designs of other moldings in which compound curves are used.



7. In architecture it is sometimes required to draw an *easement cornice* tangent to the straight cornice at B, and passing through a given point A. Explain the construction and make such a drawing.



The same construction is used in laying out the easements of stair rails.

PRACTICE TESTS

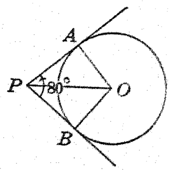
These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT FOUR

TEST ONE

Numerical Exercises

In Ex. 1–2 PA and PB are tangent to circle O:



1. If $PA = 10$ in., how long is PB? § 220.

2. If $\angle BPA = 80°$, how many degrees in $\angle AOB$? §§ 215, 133.

In Ex. 3–4 the radius of circle O is 3 in. and the radius of circle O′ is 2 in.:

3. How long is OO' if circles O and O' are tangent externally? § 221.

4. How long is OO' if circles O and O' are tangent internally? § 221.

5. Chord AB of circle O is 8 in. If $\angle AOB = 60°$, how long is OA? § 77.

6. A regular pentagon is inscribed in circle O. If AB is a side of the pentagon, how many degrees in $\angle AOB$? § 189.

7. The diameter of a circle is 10 in. What is the distance from the center of the circle to a line which is tangent to the circle? § 215.

8. In the figure for Ex. 1, if $\angle BPA = 120°$ and $PO = 20$ in., how long is PA? §§ 220, 160.

9. In circle O, arc AB = arc BC and $\angle AOB = 45°$. How large is $\angle AOC$? § 189.

10. In circle O, chord $AB = 6$ in., and chord $CD = 8$ in. One of the chords is 4 in. from the center of the circle and the other 2 in. What is the distance from O to AB? § 208.

11. In circle O the radius OA makes an angle of 30° with chord AB and of 60° with chord AC. If $AC = 10$ in., what is the distance from O to AB? § 160.

12. Triangle ABC is inscribed in a circle. $AB = 2$ in., $BC = 2\frac{1}{2}$ in., and $AC = 3$ in. Two of the central angles which have two of the sides as chords are 82° and 110°. How large is central angle AOC? § 34.

TEST TWO

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. In the same circle or in equal circles, if two chords are equal, their *arcs* are equal. § 193.

2. A *line* perpendicular to a chord bisects the chord and its arcs. § 195.

3. A circle will pass through the vertices of any right triangle if the center is the mid-point of the hypotenuse and the radius is equal to the *altitude* to the hypotenuse. § 159.

4. Two *parallel* lines intercept equal arcs on a circle. § 226.

5. A circle may be circumscribed about any *triangle*. § 184.

6. If any central angle of a circle is doubled its *chord* is doubled also. § 205.

7. *Four* of the diagonals of a regular inscribed hexagon are diameters. § 182.

8. In the same circle, equal central angles have equal *arcs*. § 189.

9. If two circles are tangent externally to each other they have *four* common tangents. § 221.

10. In the same circle or equal circles, of two unequal chords the *smaller* is nearer the center. § 208.

11. If two unequal circles are concentric, they have *no* common tangents. § 192.

12. If two chords of a circle bisect each other they are *diameters*. § 197.

TEST THREE

Multiple-Choice Statements

From the expressions printed in italics select that one which best completes the statement.

1. If the distance of a point from the center of a circle is less than the radius, the point is *at the center of the circle, equidistant from the circle, within the circle.* § 181.

2. If, in the same circle, one arc is double another arc, *the central angle of the first arc is double the central angle of the second arc, the chord of the first arc is double the chord of the second arc, the arcs are supplementary.* § 189.

3. If a circle is circumscribed about a polygon, the polygon is *equilateral, regular, inscribed in the circle.* § 198.

4. A diameter which bisects a chord (not a diameter) is *perpendicular to the chord, equal to half the chord.* § 196.

5. If the center of a circle whose radius is 5 inches, is 9 inches from the center of a circle whose radius is 4 inches, then the circles have in common *zero, one, two, three* points. § 221.

6. Tangents to a circle at the ends of a diameter are *equal, parallel, perpendicular.* §§ 215, 107.

7. Two chords of a circle equally distant from the center are *equal, parallel, perpendicular.* § 202.

8. The bisector of the angle formed by two intersecting tangents *passes through the center of the circle, is perpendicular to the radius, passes through the point of contact.* § 220.

9. The bases of a trapezoid inscribed in a circle are *equally, unequally,* distant from the center. § 208.

10. If a square and a regular pentagon are inscribed in a circle, the sides of the square are *the same distance from the center as, nearer the center than, farther from the center than* the sides of the pentagon. § 208.

11. If two major arcs of a circle are unequal, the greater major arc has the *larger, smaller* chord. § 205.

12. Two triangles have as their common base the common chord of two intersecting unequal circles. Their vertices are the centers of the circles. The triangles are *congruent, equilateral, isosceles.* *Post.* 6.

CUMULATIVE TESTS ON THE FIRST FOUR UNITS

TEST FOUR

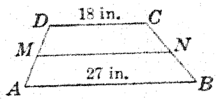
Numerical Exercises

1. The supplement of an angle is 21 degrees more than twice the angle. Find the angle. § 38.

2. The diameter of a circle is 10 inches. Its circumference is divided at A, B, C, D, E, and F into six equal arcs. How long is chord AB? §§ 189, 77.

3. The complement of a base angle of an isosceles triangle is 70°. How many degrees in the vertex angle? § 123.

4. An exterior angle of a regular polygon has 45°. How many sides has the polygon? § 134.



5. $ABCD$ is a trapezoid and M and N are the mid-points of AD and BC, respectively. How long is MN? § 157.

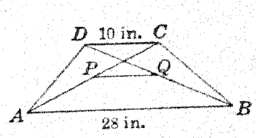
6. In the rhombus $ABCD$, diagonal AC is equal to side BC. How large is angle BCD? § 137.

7. Two tangents to a circle form an angle of 60°. Radii are drawn to the points of contact. Find the number of degrees in the angle formed by these radii. §§ 215, 133.

8. One acute angle of a right triangle is 5° less than twice the other. How many degrees in each? § 125.

9. One angle of a rhombus is 120°. If the shorter diagonal is 12 in., find a side of the rhombus. §§ 142, 77.

10. The length of the hypotenuse of a right triangle is 18 ft. Find the length of the median to the hypotenuse. § 159.



11. Each interior angle of a regular polygon is 165°. How many sides has the polygon? § 134.

12. In trapezoid $ABCD$, $AP = PC$ and $BQ = QD$. How long is PQ? § 155.

TEST FIVE

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. The diagonals of a *parallelogram* bisect the angles of the *parallelogram.* § 150, Ex. 12.

2. If the sum of the exterior angles of a polygon equals the sum of the interior angles, the polygon has *four* sides. §§ 133, 134.

3. If *two* lines are cut by a transversal, the alternate interior angles are equal. § 113.

4. Intersecting lines are *sometimes* parallel to each other. § 105.

5. If the diagonals of a quadrilateral are equal and bisect each other, the figure is a *trapezoid.* §§ 148, 162, Ex. 6.

6. If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are *equal.* § 115.

7. The consecutive angles of a *quadrilateral* are supplementary. § 115.

8. A line cutting two sides of an equilateral triangle and parallel to the third side forms an *equilateral* triangle. § 114.

9. If the diagonals of a rectangle meet at right angles, the figure is a *square.* § 150, Ex. 6.

10. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is *equilateral.* § 89.

11. The diagonal of a *parallelogram* divides it into two congruent triangles. § 142.

12. If one acute angle of a right triangle is double another, the shorter side is half the *hypotenuse.* § 160.

13. The shorter base of an isosceles trapezoid is 22 in. and the non-parallel sides are each 20 in. Since each of the equal sides makes an angle of 60° with it, the longer base is *44* in. long.

14. Three circles are drawn, each tangent externally to the other two. The segments connecting their centers are 10 in., 14 in., and 16 in., respectively. The radius of the largest circle must be *10* in.

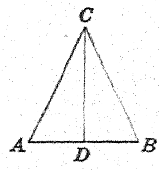
15. The segment connecting the mid-points of the diagonals of a trapezoid is 8 in. If the shorter base is 32 in., the longer base must be *40* in.

TEST SIX

Supplying Reasons

Supply axioms, postulates, or theorems as reasons for the following statements.

1. If $AC = CB$ and CD bisects angle ACB, $\triangle ACD \cong \triangle BCD$. § 64.



2. Since $\triangle ACD \cong \triangle BCD$ (Ex. 1), $\angle A = \angle B$. § 58.

3. If two straight lines intersect within a triangle, they cannot intersect outside the triangle. *Post.* 2.

4. Two right triangles are congruent if the legs of one are equal to the legs of the other. § 64.

5. In $\triangle ABC$, AD is an altitude and MX is the perpendicular bisector of side BC. **Conclusion:** $MX \parallel AD$. § 107.

6. Lines l and l' lie in the same plane and are not parallel. **Conclusion:** l and l' have a common point. § 105.

7. Given: $\angle ACX$ is an exterior angle of $\triangle ABC$. **Conclusion:** $\angle ACX = \angle A + \angle B$. § 128.

8. Given: $AB \perp BC$. **Conclusion:** AC is not $\perp$ BC. § 97.

9. Given: Quadrilateral $ABCD$ with diagonals AC and BD intersecting at O. Also $AO = OC$ and $BO = OD$. **Conclusion:** $ABCD$ is a parallelogram. § 148.

10. Given: $\triangle ABC$, M the mid-point of BC, $MN \parallel AB$ and intersects AC at N. **Conclusion:** $AN = NC$. § 153.

11. Given: Circle O, with radius 3 in. Point P is 2 in. from O. **Conclusion:** O lies inside the circle. *Post.* 13.

12. Given: $\triangle ABC$ inscribed in circle O. **Conclusion:** the perpendicular bisectors of the sides of the triangle pass through O. § 197.

TEST SEVEN

Constructions

Make the constructions, leaving all construction lines.

1. Construct a right triangle with hypotenuse $3\frac{1}{4}$ in. and side 3 in.

2. Construct a right triangle with hypotenuse 3 in. and angle 75°.

3. Construct a triangle, given the position of the mid-points of the sides.

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GEOMETRY IN A MODERN HOME

Angles, parallel lines, circles, and polygons are all to be found in the decorations and furnishings of this living room.

UNIT FIVE

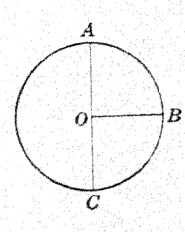
ANGLES MEASURED BY ARCS OF CIRCLES; CONCURRENT LINES; CONSTRUCTIONS

235. Measuring. In your earlier work in mathematics you have learned how to *measure* quantities and that in measuring you use a unit of the same kind as the quantity you are measuring. Thus, to measure a length you need a unit of length, such as a yardstick or a foot rule.

Thus, if the length of the foot rule is contained 9 times in a line segment, you say that the line segment is 9 feet long. So the 9 is the numerical measure of the segment when a foot is the unit of measure.

To measure an angle you similarly use as a unit a small angle, one ninetieth part of a right angle. This is called an **angle degree.**

236. Measuring arcs of circles. The unit that we use in measuring an arc of a circle is *a small arc of the same circle that we are measuring,* $\frac{1}{360}$ *part of the entire circle.* This unit is called an **arc degree.**

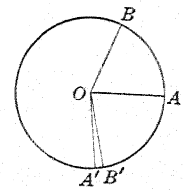


Thus, if arc AB is one fourth of the circle it contains 90 arc degrees. Similarly, if AOC is a diameter, arc ABC contains 180 arc degrees.

Just as an *angle degree* contains 60 *angle minutes* and an *angle minute* contains 60 *angle seconds,* so an arc-degree is divided into **arc minutes** and **arc seconds.**

237. Central angles and their intercepted arcs. Since an angle degree is one ninetieth of a right angle, there will be just 360 angle degrees about the point O. These 360 equal central angles will intercept 360 equal arcs (Post. 16), each of the arcs being one arc degree.

In the figure, if $\angle A'OB'$ is one angle degree, arc $A'B'$ is one arc degree. If $\angle AOB$ contains the unit angle $A'OB'$ 65 times, then arc AB contains the unit arc $A'B'$ 65 times also, and the arc AB contains 65 arc degrees.



From this reasoning it follows that an arc contains as many arc degrees as its central angle contains angle degrees. We may state:

POSTULATE 18. *A central angle has the same measure as its arc.*

238. Two arcs are **complementary** if their sum is 90 arc degrees, and two arcs are **supplementary** if their sum is 180 arc degrees.

EXERCISES

1. If there are 360 arc degrees in a circle, how many arc degrees in half a circle?

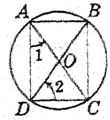
2. How many arc degrees in one-third of a circle? In one-fourth of a circle?

3. How many arc degrees in each arc of a circle circumscribed about an equilateral triangle? A square? A regular hexagon?

4. If equilateral triangle ABC is inscribed in a circle with center O, how many degrees are there in $\overset{\frown}{AB}$? In $\angle AOB$?

5. If AB is a side of an equilateral pentagon inscribed in a circle with center O, how many degrees are there in $\overset{\frown}{AB}$? In $\angle AOB$?

6. AC and BD are diameters. If $\overset{\frown}{BC} = 110°$, how large is $\overset{\frown}{AB}$? $\overset{\frown}{DC}$? $\overset{\frown}{AD}$?

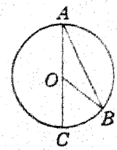


7. In Ex. 6, how large is each central angle? How large is $\angle 1$? $\angle 2$?

8. In the figure for Ex. 6, if $\overset{\frown}{AB} = 60°$, how many degrees in $\angle BDA$? In $\angle ABD$?

9. A circle is divided into five arcs so that the first is twice the second, the third one-half the second, and the fourth twice the first. If the fifth arc is 15°, how many arc degrees in each?

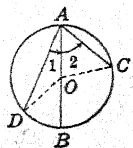
10. A flywheel makes 40 revolutions per minute. Through how many degrees of arc does a point on the rim travel in one second?



11. If AOC is a diameter and $\overset{\frown}{BC} = 50°$, how large is $\overset{\frown}{AB}$? $\angle COB$?

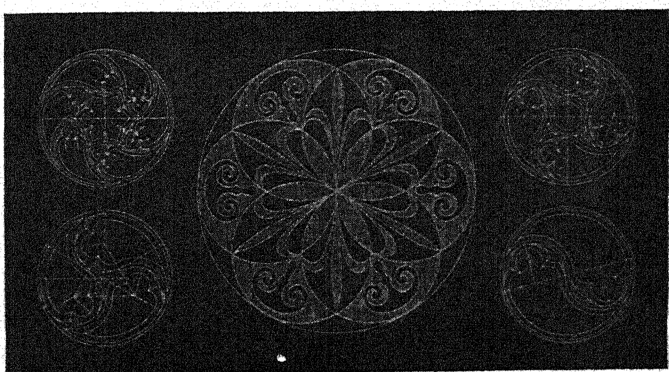
*12. In Ex. 11, how large is $\angle A$?

HINT. — Is $\triangle AOB$ isosceles? Is $\angle COB$ an exterior angle of this triangle?



*13. AOB is a diameter, $\overset{\frown}{DB} = 40°$, and $\overset{\frown}{BC} = 100°$, find (1) $\angle$s DOB and BOC and thus find (2) $\angle DAC$.

*14. In Ex. 13, how does $\angle DAC$ compare in size with $\angle DOC$?



CIRCLES USED IN ORNAMENTAL DESIGNS

239. An **inscribed** angle is an angle whose vertex lies on a circle and whose sides are chords of the circle. Angle BAC in each of the figures below is said to be **inscribed** in arc BAC. Arc BC is said to be intercepted by the angle BAC.

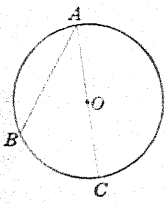


FIG. 1

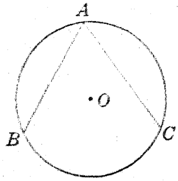


FIG. 2

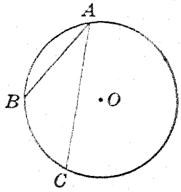


FIG. 3

240. Measuring inscribed angles. Before you read the proof of the next theorem, see if these suggestions will help you discover its proof. In Fig. 1, AC is a diameter, and $\angle BAC$ is an inscribed angle. We wish to discover the relation between the number of angle degrees in $\angle BAC$ and the number of arc degrees in $\widehat{BC}$. If radius OB is drawn, what kind of triangle is AOB? Why is $\angle BOC = 2\angle A$ (§§ 69, 128)?

If $\angle BOC = 70°$, how large is $\widehat{BC}$? $\angle A$? How large is each, if $\angle BOC = 40°$? $80°$? $x°$? Then what relation do you think there is between the number of angle degrees in an inscribed angle and the number of arc degrees in its intercepted arc? Can you prove that, when one side of an inscribed angle is a diameter, as in Fig. 1, the angle has the same measure as half its intercepted arc?

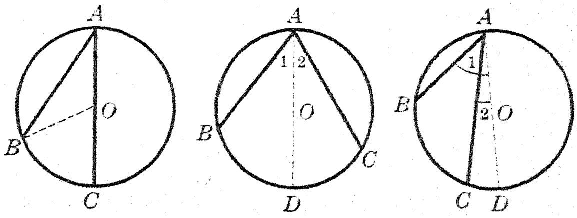
In Fig. 2, by drawing a diameter through A, can you show that $\angle BAC$ has the same measure as half of two arcs whose sum is $\widehat{BC}$?

In Fig. 3, by drawing a diameter from A, can you prove $\angle BAC$ has the same measure as half of two arcs whose difference is $\widehat{BC}$? Thus you will have proved:

An inscribed angle has the same measure as half of its intercepted arc.

PROPOSITION 1. THEOREM

241. *An inscribed angle has the same measure as half of its intercepted arc.*



Given: Angle BAC inscribed in $\odot O$.

To prove: Angle BAC has the same measure as $\frac{1}{2}\ \widehat{BC}$.

CASE I. *When the center lies on the side of the angle.*

Plan: Draw OB. Compare $\angle A$ and $\angle BOC$.

Proof:

STATEMENTS	REASONS
1. $\angle BOC$ has the same measure as $\widehat{BC}$.	1. *Post.* 18.
2. $\angle BOC = \angle A + \angle B$.	2. § 128.
3. $\angle A = \angle B$.	3. § 69.
4. $2\angle A = \angle BOC$, and hence, $2\angle A$ has the same measure as $\widehat{BC}$.	4. *Ax.* 7.
5. $\angle A$ has the same measure as $\frac{1}{2}\ \widehat{BC}$.	5. *Ax.* 5.

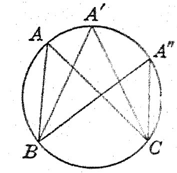
CASE II. *When the center lies within the angle.*

Plan: Draw diameter AD. Apply Case I to $\angle BAD$ and to $\angle DAC$ and add.

CASE III. *When the center is outside of the angle.*

Plan: Draw diameter AD. Apply Case I to $\angle BAD$ and to $\angle CAD$ and subtract.

242. Corollary 1. *Angles inscribed in the same arc or in equal arcs are equal.*

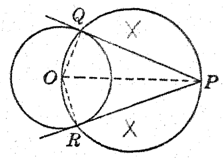


Plan: Show that each angle has the same measure as $\frac{1}{2}\overset{\frown}{BC}$.

243. Corollary 2. *An angle inscribed in a semicircle is a right angle.*

Construction XV

244. *To construct a tangent to a circle from a given outside point.*



Given: Circle O; point P outside the $\odot$.

Required: Construct a tangent to the $\odot$ from P.

Plan: Draw a circle having OP as diameter, intersecting $\odot\ O$ in Q and R. Draw PQ and PR. PQ and PR are the required tangents.

Proof: Think: "Since OQP is a semicircle, $\angle Q$ is . . ." (§ 243). Then use § 214.

Write the construction and proof.

EXERCISES

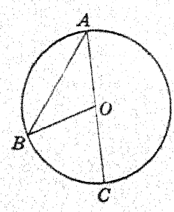


Fig. 1

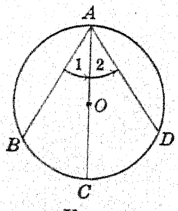


Fig. 2

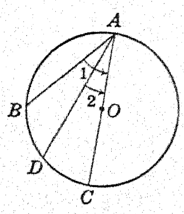
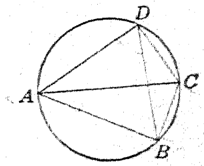


Fig. 3

1. In Fig. 1, how many degrees in $\angle A$ if $\angle BOC = 60°$? If $\overset{\frown}{BC} = 70°$? If $\overset{\frown}{AB} = 124°$?

2. In Fig. 2, how many degrees in $\angle BAD$ if $\overset{\frown}{AB} = 110°$ and $\overset{\frown}{AD} = 100°$? If $\angle 1 = 40°$ and $\overset{\frown}{CD} = 45°$? If $\overset{\frown}{BC} = 54°$ and $\overset{\frown}{AD} = 140°$?

3. In Fig. 3, how many degrees in $\angle BAD$ if $\overset{\frown}{DCA} = 230°$ and $\angle 1 = 80°$? If $\overset{\frown}{AB} = 50°$, and $\overset{\frown}{DC} = 60°$?



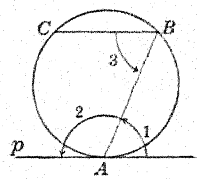
4. If $\overset{\frown}{AB} = 110°$, find $\angle ADB$; $\angle ACB$.

5. If $\overset{\frown}{BC} = 58°$, find $\angle BAC$; $\angle BDC$.

6. If $\overset{\frown}{CD} = 70°$, find $\angle CAD$; $\angle CBD$.

7. If $\overset{\frown}{DA} = 122°$, find $\angle DBA$; $\angle DCA$.

8. Find the angles in Ex. 4–7 if $\overset{\frown}{AB} = 100°$, $\overset{\frown}{BC} = 50°$, and $\overset{\frown}{CD} = 60°$.



9. Line p is tangent to the circle at A, and AB and CB are chords; $CB \parallel p$. If $\overset{\frown}{AB} = 128°$, how large is $\overset{\frown}{CA}$? $\angle 3$? $\angle 1$? $\angle 2$?

10. Two sides of an inscribed triangle have arcs of 125° and 140°. How many degrees in each angle of the triangle?

---

11. Prove that an angle inscribed in an arc that is less than a semicircle is an obtuse angle.

12. Prove that an angle inscribed in an arc that is greater than a semicircle is an acute angle.

13. If chords AB and CD intersect within a circle in a point E, prove that $\triangle ACE$ and $\triangle BDE$ are mutually equiangular.

NOTE. — If two triangles have the angles of one respectively equal to the angles of the other, the triangles are said to be **mutually equiangular.**

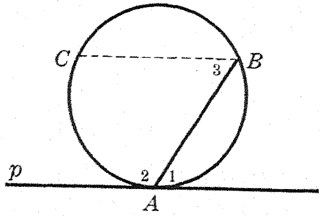
14. Prove that the opposite angles of a quadrilateral inscribed in a circle are supplementary.

***15.** In the figure for Ex. 9, tangent p is parallel to chord CB. Why is $\angle 1 = \angle 3$? Why is $\overset{\frown}{AB} = \overset{\frown}{CA}$?

***16.** Using the results obtained in Ex. 15, prove that $\angle 1$ has the same measure as $\frac{1}{2}\, \overset{\frown}{AB}$.

PROPOSITION 2. THEOREM

245. *An angle formed by a tangent and a chord from the point of contact has the same measure as half its intercepted arc.*



Given: A circle with p tangent at A, and chord AB.

To prove: $\angle 1$ has the same measure as $\frac{1}{2}\,\widehat{AB}$, and $\angle 2$ has the same measure as $\frac{1}{2}\,\widehat{ACB}$.

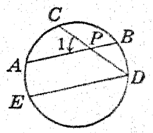
Analysis: If $BC \parallel p$, $\angle 3$ will equal $\angle 1$ (§ 113). What is the measure of $\angle 3$? (§ 241). Why is $\widehat{AC} = \widehat{AB}$?

Proof: Left for you to write.

EXERCISES

1. In the figure for § 245, how many degrees in $\angle 1$ if $\widehat{AC}$ is 110°?

2. Chords AB and CD intersect at P and chord $DE \parallel$ chord AB. If $\widehat{BD} = 36°$ and $\widehat{AC} = 70°$, how large is $\angle CDE$?



3. Using the data given in Ex. 2, how large is $\angle 1$?

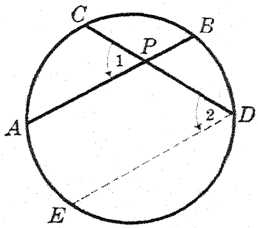
4. In the same figure, find $\angle 1$ if $\widehat{BD} = 40°$, and $\widehat{AC} = 80°$. If $\widehat{BD} = 50°$, and $\widehat{AC} = 90°$.

---

5. From the results obtained in Ex. 2–4, see if you can write a theorem about the measure of the angle formed by two intersecting chords.

Proposition 3. Theorem

246. *An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.*



Given: Circle O, chords AB and CD intersecting at P.

To prove: $\angle CPA$ has the same measure as $\frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$.

Plan: Draw $DE \parallel AB$. (1) $\angle 2$ has the same measure as . . . ? (2) $\overset{\frown}{AE} = \overset{\frown}{BD}$. Why?

Proof: Write the proof in full.

EXERCISES

1. In the figure of § 246, how many degrees in $\angle 1$ if $\overset{\frown}{AC} = 70°$ and $\overset{\frown}{AE} = 60°$?

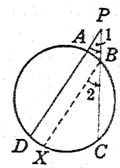
2. In the same figure, if $\overset{\frown}{AC}$ is a quadrant and $\angle 1$ is 70°, how many degrees in $\overset{\frown}{BD}$?

3. Lines are drawn on the face of a clock from 12 to 4 to 7 to 10 to 2. How many degrees in each arc? How many degrees in each angle formed?

4. Would the theorem in § 246 be true if P were at the center of the circle? To what fundamental theorem would it reduce?

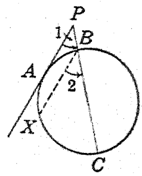
5. To what theorem would it reduce if P falls on the circle?

6. Angle DPC is formed by two secants intersecting the circle as shown. If $\widehat{BA} = 20°$, $\widehat{DC} = 72°$, and $BX \parallel AD$, how large is $\widehat{DX}$? $\widehat{XC}$? $\angle 2$? $\angle 1$?



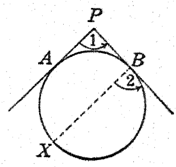
7. In the figure for Ex. 6, how large is $\angle 1$, if $\widehat{BA} = 30°$ and $\widehat{DC} = 80°$?

8. PA is tangent at point A and secant PC intersects the circle at points B and C, respectively. If $\widehat{BA} = 30°$, $\widehat{AC} = 70°$, and $BX \parallel PA$, how large is $\widehat{AX}$? $\widehat{XC}$? $\angle 2$? $\angle 1$?



9. In the figure for Ex. 8, how large is $\angle 1$, if $\widehat{BA} = 40°$ and $\widehat{AC} = 84°$?

10. PA and PB are tangent at points A and B, respectively. If $\widehat{BA} = 64°$, $\widehat{AXB} = 296°$, and $BX \parallel PA$, how large is $\widehat{AX}$? $\widehat{XB}$? $\angle 2$? $\angle 1$?



11. In the figure for Ex. 10, how large is $\angle 1$, if $\widehat{BA} = 50°$ and $\widehat{AXB} = 310°$?

---

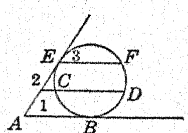
12. From the results obtained in Ex. 6 and 7 form a theorem about the measure of an angle formed by two secants.

13. From the results obtained in Ex. 8 and 9 form a theorem about the measure of an angle formed by a tangent and a secant.

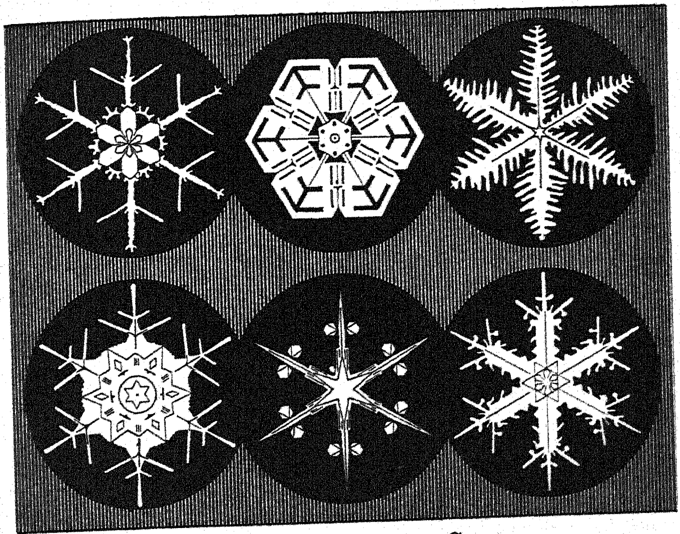
14. From the results obtained in Ex. 10 and 11 form a theorem about the measure of an angle formed by two tangents.

15. If two secants intersect within a circle, what is the measure of the angle formed? What is the measure of the angle if they intersect at the center of a circle? On a circle? Outside a circle?

***16.** AB and AE are tangent to the circle at B and E, respectively, and $AB \parallel CD \parallel EF$. Why are angles 1, 2, and 3 equal? Why has $\angle 3$ the same measure as $\frac{1}{2}\,\widehat{EF}$? Can you prove that $\angle 2$ has the same measure as $\frac{1}{2}\,(\widehat{EFD} - \widehat{EC})$?



***17.** In the figure of Ex. 16 prove that $\angle 1$ has the same measure as $\frac{1}{2}\,(\widehat{EFDB} - \widehat{EB})$.



GEOMETRIC FORM IN SNOW CRYSTALS

These snow crystals represent only a few of the hundreds of forms. Manufacturers use such photographs as a basis for jewelry designs.

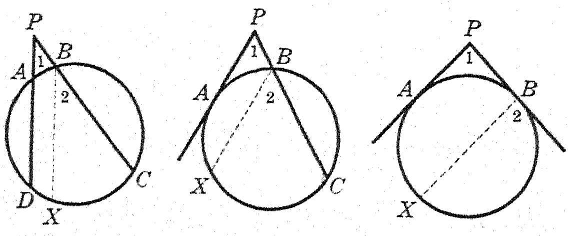
247. Angles formed by tangents and secants. If two secants intersect outside a circle, as in the figure for Ex. 6, and if BX is drawn $\parallel PD$, $\angle 1$ is equal to what angle? $\angle 2$ has the same measure as half of what arc? Is $\widehat{CX} = \widehat{CD} - \widehat{XD}$? Why is $\widehat{DX} = \widehat{BA}$? See if you can prove that $\angle P$ has the same measure as $\frac{1}{2}(\widehat{DXC} - \widehat{BA})$.

If PC is held fixed, and PD revolves about P until it is tangent at A (see figure for Ex. 8), do you think $\angle P$ has the same measure as $\frac{1}{2}(\widehat{AC} - \widehat{AB})$? See if you can prove it.

If PA is held fixed and PC revolves until it is tangent at B (see figure for Ex. 10), prove $\angle P$ has the same measure as $\frac{1}{2}(\widehat{AXB} - \widehat{AB})$.

Proposition 4. Theorem

248. *An angle formed by two secants, by a secant and a tangent, or by two tangents intersecting outside the circle has the same measure as half the difference between the intercepted arcs.*



Case I. **Given:** Circle with secants PAD and PBC intersecting at P outside the $\odot$.

To prove: $\angle P$ is measured by $\frac{1}{2}(\widehat{CD} - \widehat{AB})$.

Plan: Construct $BX \parallel PD$. Then compare $\measuredangle$ 1 and 2. Observe that $\widehat{DX} = \widehat{AB}$.

Proof:

STATEMENTS	REASONS
1. Draw $BX \parallel PD$.	1. § 122.
2. $\widehat{CD} = \widehat{CX} + \widehat{XD}$ and $\widehat{CX} = \widehat{CD} - \widehat{XD}$.	2. *Why?*
3. $\angle 2$ has the same measure as $\frac{1}{2}\widehat{CX}$.	3. § 241.
4. $\angle 1 = \angle 2$.	4. § 114.
5. $\therefore \angle 1$ has the same measure as $\frac{1}{2}(\widehat{CD} - \widehat{XD})$.	5. *Ax.* 7.
6. $\widehat{XD} = \widehat{AB}$.	6. § 226.
7. $\therefore \angle P$ has the same measure as $\frac{1}{2}(\widehat{CD} - \widehat{AB})$.	7. *Ax.* 7.

Case II. **Given:** Circle O, with tangent PA and secant PBC, intersecting at P, outside the $\odot$.

To prove: $\angle P$ is measured by $\frac{1}{2}(\widehat{AC} - \widehat{AB})$.

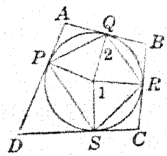
Proof: Similar to I. Write in full.

Case III. **Given:** Circle O, with tangents PA and PB, intersecting at P.

To prove: $\angle P$ is measured by $\frac{1}{2}(\widehat{AXB} - \widehat{AB})$.

The proofs of Cases II and III are similar to that of Case I. Write in full.

EXERCISES

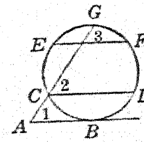


1. $ABCD$ is a circumscribed and $PQRS$ an inscribed quadrilateral. P, Q, R, and S are points of tangency. If the vertices of $PQRS$ are joined to the center of the circle, as shown, and if $\widehat{PQ} = 80°$, $\widehat{QR} = 70°$, and $\angle 1 = 85°$, find the number of degrees in: $\widehat{RS}$; $\widehat{PS}$; $\angle 2$.

2. Find the number of degrees in each angle of $PQRS$.

3. Find the number of degrees in each angle of $ABCD$.

4. How large is $\angle PQA$?



5. If $AB \parallel CD \parallel EF$, AB is tangent at B, ACG is a secant, $\widehat{GF} = 40°$, $\widehat{BD} = 45°$, and $\angle 1 = 55°$, how many degrees in $\widehat{FD}$?

6. In Ex. 5, how many degrees in $\widehat{EG}$?

7. Two equal chords AB and AC form an inscribed angle of $70°$. How many degrees in arc BC? In arc AB? In arc AC?

---

8. Is the chord that joins the points of contact of two parallel tangents a diameter?

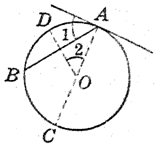
9. If chords AB and AC make equal angles with the tangent at A, prove that $AB = AC$.

10. Prove that the chords which connect the ends of two intersecting diameters form a rectangle.

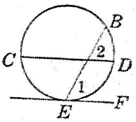
11. If $\widehat{AC}$ is bisected at point B, $\triangle ABC$ is isosceles.

12. If two chords are perpendicular to each other, the sum of one pair of arcs intercepted by vertical angles equals the sum of the other pair, and each sum equals a semicircle.

13. Prove the theorem in § 245 by drawing a radius perpendicular to the chord and also the diameter from the point of contact of the tangent. ($\angle 1 = \angle 2$. Why?)



14. Using this figure, prove the theorem in § 245. Draw $CD \parallel EF$ and intersecting EB.



15. If an angle inscribed in an arc of a circle is a right angle, the arc is a semicircle.

16. If an angle inscribed in an arc is obtuse, the arc is less than a semicircle.

17. If an angle inscribed in an arc is acute, the arc is greater than a semicircle.

249. A. Continuity. It is interesting to note how the statements in §§ 237–248 can be combined into a single statement about the measure of an angle formed by two intersecting lines.

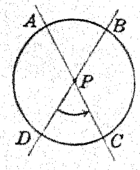


FIG. 1

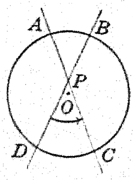


FIG. 2

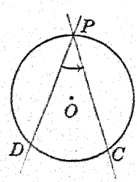


FIG. 3

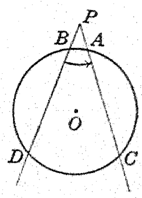


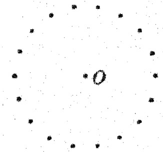
FIG. 4

We must first define *negative* and *positive* arcs. If from the point P an arc appears *concave* (like arc CD) we shall call it a **positive arc**; if it appears *convex* (like arc BA in Fig. 4), we shall call it a **negative arc**. Then in each case *the angle is measured by half the algebraic sum of the inter-*

cepted arcs. In Fig. 1, P is at the center of the circle and angle P is measured by $\frac{1}{2}(\widehat{AB} + \widehat{CD})$. Since $\widehat{AB} = \widehat{CD}$ (Post. 16), angle P is measured by $\widehat{CD}$ as in Post. 18. In Fig. 3, $\widehat{AB}$ is zero; in Fig. 4, $\widehat{AB}$ is negative. Similarly draw a figure for each of the theorems in §§ 241–248.

Locus

250. Finding a locus. The following exercises will show you how to locate all possible positions of a point which satisfy certain conditions.



1. Where are all points which are 1 in. from O?

Here the condition is that each point must be 1 in. from O. There are an infinitely large number of points which satisfy this condition.

Placing a number of points 1 in. from O suggests two things:

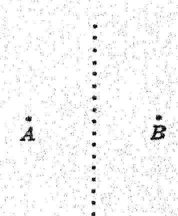
a. All points 1 in. from O lie on the circle indicated.
b. All points on the circle are 1 in. from O.

The problem is usually stated:

What is the locus of points 1 in. from O? The answer is:

The locus of points 1 in. from O is a circle with center O and radius 1 in.

NOTE. — The plural of locus is *loci* (pronounced lō-sī).



2. What is the locus of points equidistant from the points A and B?

After locating a few points, each of which satisfies the condition that it is equidistant from A and B it is evident that all such points lie on the perpendicular bisector of the segment AB. The answer is stated:

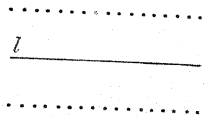
The locus of points equidistant from A and B is the perpendicular bisector of AB.

This is true because

a. All points equidistant from A and B lie on the perpendicular bisector of AB (§ 89*a*).

b. All points on the perpendicular bisector of AB are equidistant from A and B (§ 89*b*).

3. What is the locus of points $\frac{1}{4}$ in. from a line l?



Answer: *The locus of points $\frac{1}{4}$ in. from a line l is a pair of lines parallel to l and $\frac{1}{4}$ in. from it.*

The first step in solving a locus problem is to select a sufficient number of points to clearly indicate the form of the locus. The second step is to state accurately what the locus is.

EXERCISES

1. Draw two parallel lines AB and CD 2 inches apart. Mark quite a number of points, each of which is 1 inch from both lines. What is the locus of points 1 in. from AB and CD?

2. Draw two lines intersecting at a point P and forming four angles. Within one of the angles mark a point which seems to be equidistant from the sides of that angle. Locate about eight other points in the same way within that angle. Repeat the process for each of the four angles. Designate the lines formed as the *locus.* Could one of these lines alone be considered as the locus of points equidistant from two intersecting lines? Would it contain *all* the points equidistant from both lines?

3. Does a semicircle contain *all* points equidistant from its center? Is it the locus of such points? Why?

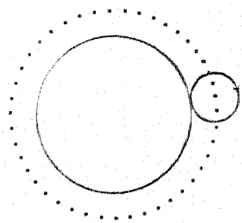
4. Draw the locus of points 1 in. from a given line l. How many lines form your locus?

5. Find experimentally the locus of the vertices of isosceles triangles having a given base AB. What does the locus seem to be?

6. Draw a square with side 3 in. Find the locus of points inside the square whose distance from a side of the square is 1 in. How many lines form the locus?

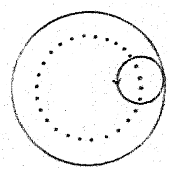
7. A wheel rolls along a level road. Represent the road by a line and draw the wheel in a number of positions, marking its center each time. What is the locus of its center?

8. Draw two parallel lines l and l', 3 in. apart. Find the locus of points each of which is 1 in. from l and 2 in. from l'.



9. A small circle rolls on the outside of a larger circle. Draw the position of the small circle a number of times, each time marking the center. State what the locus of the center seems to be.

10. Take any point P outside a given line l and draw 8 or 10 segments from P and terminated by l. Estimate the center of each segment and mark it. What does the locus of these mid-points seem to be?



11. What does the locus of the center of the small circle seem to be as it rolls inside the large circle? Draw it.

12. Draw a triangle ABC with its base AB 2 in. and the altitude to AB 1 in. Locate another point C' which is 1 in. from AB, and draw $C'A$ and $C'B$. In a like manner locate six or eight points and connect them to A and B. What, then, is the locus of the vertices of triangles having a given base AB of 2 in. and a given altitude of 1 in.? Are there any such points below AB? (*All* points must be indicated to have the locus.)

*13. Draw a circle with a radius of 2 in. Take a point P about 1 in. outside the circle. Now draw about 10 or 12 segments from P and terminated by the farther arc of the circle, and mark the mid-point of each. Also mark the mid-point of each segment from P to the nearer arc of the circle. What does the locus seem to be?

*14. Draw a right triangle with its hypotenuse AB and C the vertex of the right angle. How many right triangles can you draw having hypotenuse AB? Carefully draw eight, all on the same hypotenuse AB and mark the positions of C. Is the locus of C a straight line?

*15. A bridge is to be constructed over a river so as to be the same distance from two towns A and B which are 10 miles apart and on opposite sides of the river. If the distance from A to the river is 2 miles, and from B to the river is 5 miles, draw a sketch and show how to locate the bridge.

251. The proof of a locus problem. You have seen that

All the points, and only those points which fulfill a certain requirement, form a figure which is called a locus.

Hence you see that, to prove the correctness of a statement about a locus, you must prove two things:

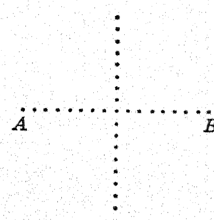
1. *All points on the locus satisfy the requirements.*
2. *All points which satisfy the requirements are on the locus.*

Examples in the next sections show you how this is done.

252. A circle as a locus. From the definition of a circle we have the:

Locus theorem I: *The locus of points at a given distance from a given point is a circle with the given point as center and with the given distance as radius.*

253. Problem: Find the locus of points equidistant from two given points.

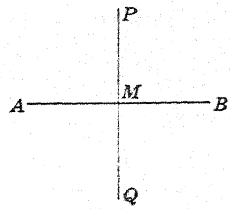


Given: Points A and B.

Required: The locus of points equidistant from A and B.

Plan: Locate a number of points each equidistant from A and B. These suggest the perpendicular bisector of the segment connecting A and B. It must now be proved that this is correct. Hence

254. Locus theorem II: *The locus of points equidistant from two given points is the perpendicular bisector of the segment connecting the points.*



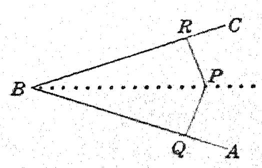
Given: PQ the perpendicular bisector of AB.

To prove: PQ is the locus of points equidistant from A and B.

Proof:

STATEMENTS	REASONS
1. All points on PQ are equidistant from A and B.	1. § 89*a*.
2. All points equidistant from A and B are on PQ.	2. § 89*b*.
3. PQ is the required locus.	3. § 251.

255. Problem: Find the locus of points equidistant from the sides of an angle.

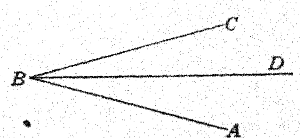


Given: Angle ABC.

Required: The locus of points equidistant from AB and CB.

Plan: Locate a number of points each of which is equidistant from AB and CB (for example, $\perp PQ = \perp PR$). These suggest the bisector of angle ABC. It must be proved that this is correct.

256. Locus theorem III: *The locus of points equidistant from the sides of an angle is the bisector of the angle.*



Given: DB the bisector of $\angle ABC$.

To prove: DB is the locus of points equidistant from AB and CB.

Proof:

STATEMENTS	REASONS
1. All points on DB are equidistant from AB and CB.	1. § 100*a*.
2. All points equidistant from AB and CB are on DB.	2. § 100*b*.
3. DB is the required locus.	3. § 251.

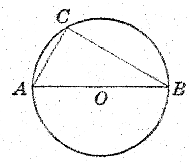
257. The theorem in § 256 may also be stated:

The locus of points equidistant from two intersecting straight lines is the pair of lines bisecting the angles formed.

258. Locus theorem IV: *The locus of points equidistant from two parallel lines is the line parallel to each of them and midway between them.*

259. Locus theorem V: *The locus of points at a given distance from a given line is a pair of lines, one on either side of the given line, each parallel to the given line, and at the given distance from it.*

260. Locus theorem VI: *The locus of the vertex of the right angle of a right triangle having a given hypotenuse is a circle having the given hypotenuse as diameter.*



Given: Circle O having AB as diameter, right $\triangle ABC$, $\angle C$ a right angle.

To prove: Circle O is the locus of C.

Proof:

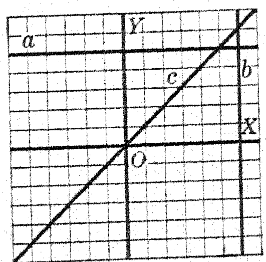
STATEMENTS	REASONS
1. Any point C on circle O is the vertex of a right triangle having AB as hypotenuse.	1. § 243.

STATEMENTS	REASONS
2. The vertex C of any right triangle having AB as hypotenuse is the circle O; for, if C were outside the circle, $\angle C$ would be acute; if C were inside the circle, $\angle C$ would be obtuse.	2. §§ 246, 248.

261. A. Path of a moving point. It is sometimes convenient to think of a locus as *the path of a point which moves so that it fulfills certain requirements.*

Thus, if a point moves so that its distance from a point O is always 3 in., its locus (or path) is a circle with center O and radius 3 in.

262. A. Locus in algebra. If a point moves so that its distance from the x-axis is always $+5$ units, its locus is represented by the line a whose equation is $y = 5$. If the distance from the y-axis is always $+6$ units, the locus of the point is the line b. Its equation is $x = 6$. Line c shows the locus of a point which moves so that its *abscissa* is always equal to its *ordinate*. The equation is $x = y$.



NOTE. — The *abscissa* is the distance from the y-axis; the *ordinate* is the distance from the x-axis.

EXERCISES

Find the locus of points which move as indicated. Write the equation and draw the locus.

1. A point moves so that it is always 10 units to the left of the y-axis.

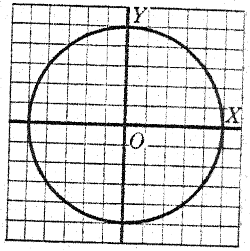
2. A point moves so that it is always 8 units below the x-axis.

3. The abscissa is always equal to the negative of the ordinate.

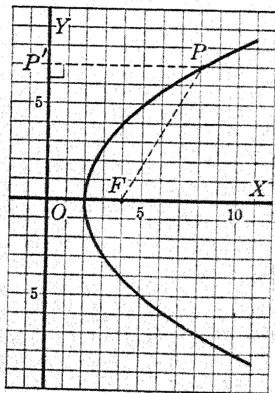
4. The ordinate is equal to three times the abscissa.

5. The ordinate is 5 more than twice the abscissa.

263. A. **Loci with second degree equations.** The loci in the preceding section were all represented by equations of the first degree in x and y. Some interesting curves are represented by second degree equations. You are not prepared yet to derive the equations of these loci, but you will study about them in later courses in mathematics.

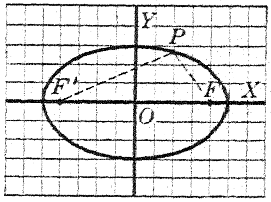


Circle. The locus of a point which moves so that its distance from a fixed point is a constant is a **circle.** If the *origin* is taken as center and r is the radius the equation is $x^2 + y^2 = r^2$. In the figure $r = 5$ and $x^2 + y^2 = 25$.

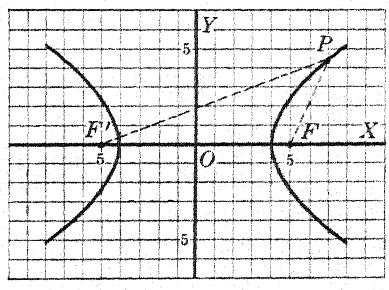


Parabola. The locus of a point which moves so that its distances from a fixed point and from a fixed line are equal is a **parabola.** In the figure the fixed point is the point $F(x = 4, y = 0)$. The fixed line is the y-axis. Thus, if P is the moving point, $PF = PP'$. The equation is $y^2 = 8\ (x - 2)$. A rifle bullet or any thrown object follows a path which is approximately a parabola.

Ellipse. The locus of a point which moves so that the sum of its distances from two fixed points is a constant is an **ellipse.** In the figure the fixed points are the points $F(x = 4, y = 0)$ and $F'(x = -4, y = 0)$ and the constant is 10. Thus, if P is the moving point, $PF + PF' = 10$. The equation is $9x^2 + 25y^2 = 225$.



Hyperbola. The locus of a point which moves so that the difference of its distances from two fixed points is a constant is a **hyperbola.** In the figure the fixed points are the points $F(x = 5, y = 0)$ and $F'(x = -5, y = 0)$. The constant is 8. Thus if P is the moving point, $PF' - PF = 8$. The equation is $9x^2 - 16y^2 = 144$.



Conic Sections. The circle, ellipse, parabola, and hyperbola are called conic sections because they are the curves formed when a plane cuts the surface of a cone. They have many interesting properties.

The points F and F' are called foci (fō-sī). If an automobile headlight is parabolic, a light placed at the focus (F) will reflect all rays in a direction parallel to the x-axis. Similarly rays of light from a distant source of light (a star for example) will, after reflection from a parabolic mirror, be brought to a focus at F. This is the principle of refracting telescopes.

The earth and the planets move about the sun in elliptical paths or orbits. The sun is at one of the foci.

If a room has an elliptical form and the walls are made to reflect sound, a whisper made at one of the foci can be heard at the other. This is the principle of the *whispering gallery.*

264. A. Linkages. The illustrations show devices for imposing requirements on moving points. Such devices are called **linkages.** Fig. 1 represents the method James Watt, the inventor of the steam engine, used to make a point trace a straight line. Parts of the "figure 8" curve that the linkage traces are

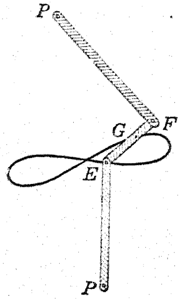


FIG. 1

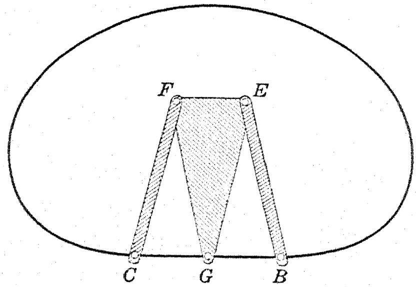


FIG. 2

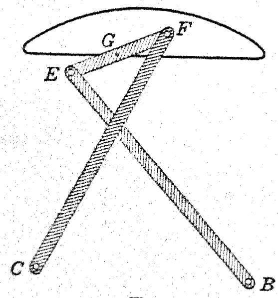


FIG. 3

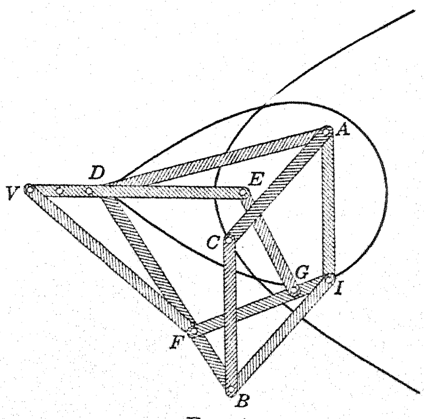


FIG. 4

nearly straight. In this linkage the longer links are of equal length and are pivoted at points P. Point G, the mid-point of EF, traces the curve. In Figs. 2 and 3, point G traces a "loaf of bread" curve. Points C and B are fixed. While point I in Fig. 4 traces a pear-shaped curve, point C traces a parabola.

265. A. Other loci. There are many other interesting loci.

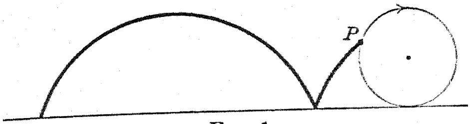


FIG. 1

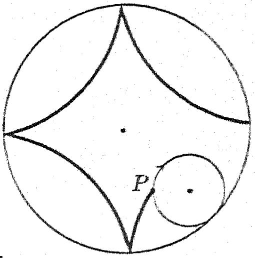


FIG. 2

CYCLOID

Thus the locus of a fixed point on a circle as the circle rolls along a straight line is the *cycloid* shown (Fig. 1). You can see it by marking a point on a hoop and then rolling the hoop along the floor or blackboard rail. If the circle rolls on the inside of another circle the locus is a *hypocycloid* (Fig. 2). (In the figure the radius of the rolling circle is one fourth that of the fixed circle. In this case the curve is also called an *asteroid*.)

EXERCISES

1. What is the locus of the tip of the minute hand of a clock?
2. What theorem proves your statement in Ex. 1?
3. What is the locus of all cities that are five miles from a given railroad?
4. Which of the locus theorems proves your statement in Ex. 3 if the railroad is straight?
5. A man rows up a stream keeping the same distance from either shore. What is the locus of the boat?
6. If the river banks are parallel, what theorem proves the answer in Ex. 5?
7. What is the locus of all houses one mile from your school? Why?

---

8. The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by the lines. Show how this theorem follows from § 256.

9. Find experimentally the locus of points which lie inside a circle and are equidistant from two points on the circle.

Determine experimentally the following loci. No proof is required.

10. A dime rolls around a half dollar, the edges of the coins always touching. What is the locus of the center of the dime?

11. A six-inch ruler moves so that its ends always are on two adjacent edges of your paper. Find the locus of its mid-point.

12. Draw a circle with a diameter of 5 in. Mark a three-inch segment AB on the edge of a piece of paper and mark its mid-point M. If the segment moves so that A and B always lie on the circle, indicate by a dotted line the locus of M. What does the locus seem to be?

13. Determine the locus of points 2 in. from a circle of radius 5 in. Be sure you have the complete locus.

14. Determine the locus of the mid-points of radii drawn in a given circle. Can you prove it?

15. Segments terminated by two sides of a triangle are parallel to the base. Determine the locus of their mid-points.

16. Segments terminated by the non-parallel sides of a trapezoid are parallel to the bases. Determine the locus of their mid-points.

17. Determine the locus of the vertex of a triangle whose base is 3 in. and whose median drawn to that base is 2 in.

266. Intersection of loci. It is possible to have a point or points which lie on two loci.

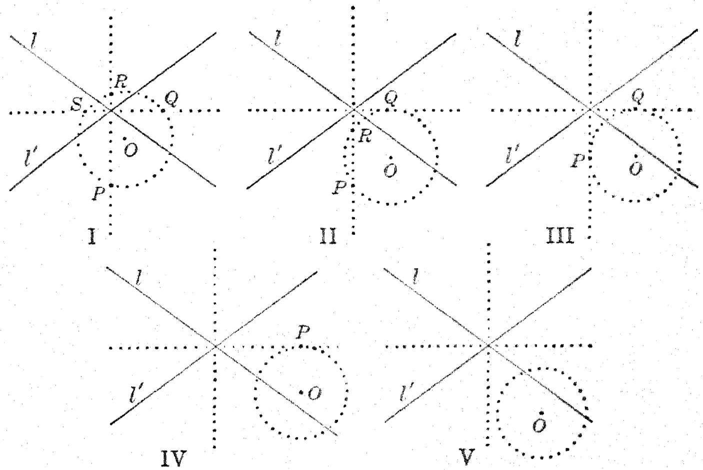
EXAMPLE. — Find the locus of points equidistant from two given intersecting lines and also $\frac{1}{2}$ in. from a given point.

Given: Intersecting lines, l and l' and point O.

Required: The locus of points equidistant from l and l' and $\frac{1}{2}$ in. from O.

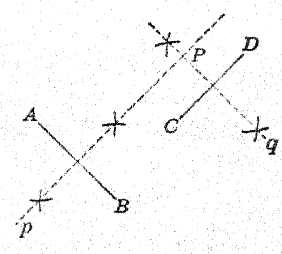
SOLUTION. — From § 257 you know that the locus of points equidistant from the two intersecting lines l and l' is the pair of lines shown in each of the figures. From § 252 you know that the locus of points $\frac{1}{2}$ in. from the point O is the circle with O as center and radius $\frac{1}{2}$ in. These circles are indicated

in each figure. *In general* the two loci intersect in the four points P, Q, R, and S shown in Fig. I. The other figures



show that there may be (dependent on the relative positions of l, l', and O) three (Fig. II), two (Fig. III), one (Fig. IV), or no (Fig. V) points common to the two loci.

EXERCISES



1. What is the locus of points equidistant from points A and B and also equidistant from points C and D?

2. A segment AB is 2 in. long. A point is equidistant from A and B and is also 3 in. from a point C. Find the possible positions of the point. When is there no solution? One solution? Two solutions? Can there be more than two solutions?

3. Find the locus of a point equidistant from the sides of an angle and also 2 in. from a given point P. When are there two solutions? One solution? No solution?

4. Find the locus of a point equidistant from the sides of an angle and equidistant from two given points. Can there be no solution?

5. Find the locus of a point equidistant from two given parallel lines and also equidistant from the sides of a given angle.

6. Given three points A, B, and C. What is the locus of points equidistant from A, B, and C? Will there always be a locus?

7. Pirates buried treasure 60 feet from a certain tree and 80 feet from a certain straight path. Show how to locate the treasure. When are there two possible locations? Only one? More than two?

8. If the treasure had been buried equidistant from a certain oak and a certain maple and 40 yds. from a certain straight road, show how to locate it.

---

9. In a given line, find all points equidistant from two given points.

10. Find all points on a given circle equidistant from two given points.

11. Find all points which lie on a given line and are equidistant from two given intersecting lines.

12. What is the locus of points 2 in. from a given line and equidistant from two parallel lines?

13. What is the locus of points equidistant from two intersecting lines and 3 in. from a given point?

14. What is the locus of points equidistant from two parallels and also equidistant from two intersecting lines?

15. What is the locus of points equidistant from two given points and also at a given distance from a given line?

267. A. Complete solution and proof of locus problems. The following steps are necessary to solve a locus problem:

1. **Locate experimentally a sufficient number of points to indicate the kind of a figure formed by the locus.**
2. **Accurately state this in the form of a locus theorem.**
3. **Prove that your statement is correct by proving:**
   a. All points on the locus satisfy the requirements, and
   b. All points which satisfy the requirements are on the locus or by proving:

c. All points on the locus satisfy the requirements, and

b. A point which does not satisfy the requirements does not lie on the locus.

A locus problem can frequently be proved by stating one of the fundamental locus theorems.

EXERCISES

(Optional)

1. The diagonals of a rectangle are 4 in. long. A fixed point O is their intersection. What is the locus of the vertices of the rectangle?

2. What is the locus of the mid-points of all chords drawn through a fixed point A on a circle whose center is O?

Suggestion. — Draw through A a number of chords, including the diameter. Locate accurately the mid-point of each. Can you prove that the locus is a circle with AO as diameter? Use § 196 and § 260.

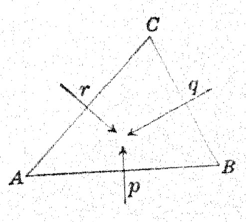
3. What is the locus of the mid-points of all chords drawn through a fixed point P within a circle with center O?

Suggestion. — Draw a number of chords through P, including the diameter and the chord perpendicular to the diameter. Is the locus a circle with diameter OP?

Theorems about Concurrent Lines

268. Three or more lines which pass through the same point are said to be **concurrent.**

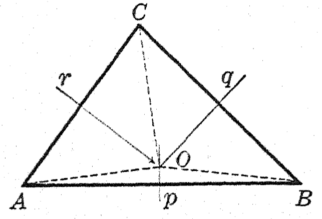
269. The perpendicular bisectors of the sides of a triangle. Why will the perpendicular bisectors of two sides of $\triangle ABC$, such as p and q, meet (Post. 15)?



If p and q meet at a point O, what can you say about the distances of O from A and B? From B and C? From A and C? Then why will r pass through O?

PROPOSITION 5. THEOREM

270. B. *The perpendicular bisectors of the sides of a triangle are concurrent.*



Given: Triangle ABC, with p, q, and r perpendicular bisectors of the sides.

To prove: p, q, and r are concurrent.

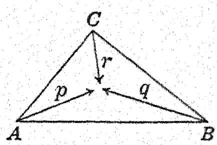
Plan: See § 184.

Proof: Write the proof.

271. Circumcenter. The point of intersection of the perpendicular bisectors of the sides of a triangle is called the **circumcenter** of the triangle. It is the center of the circumscribed circle.

Ex. 1. If a polygon is inscribed in a circle, are the perpendicular bisectors of its sides concurrent? What is the common point of intersection?

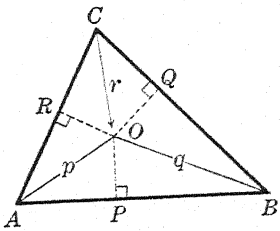
272. The bisectors of the angles of a triangle. Why will two of the bisectors, such as p and q, intersect, say in a point O? How do we find the distance from O to the sides of the triangle (§ 99)? Are these distances equal (§ 256)? Then why will r pass through O?



Can you prove that the bisectors of the angles of a triangle are concurrent?

PROPOSITION 6. THEOREM

273. B. *The bisectors of the angles of a triangle are concurrent.*



Given: Triangle ABC, with p, q, and r bisecting angles A, B, and C, respectively.

To prove: p, q, and r concurrent.

Plan: Show that any two angle bisectors, such as p and q, intersect in a point O. Then show that, since the perpendicular distances OP, OQ, and OR are all equal, O lies on r.

Proof: Write the proof.

EXERCISES

1. Draw a triangle ABC with $AB = 2\frac{1}{2}$ in., $AC = 3$ in., and $BC = 3\frac{1}{2}$ in. Construct the bisectors of angles A, B, and C, and, as in the figure of § 273, call the point of intersection O. With O as a center, and perpendicular distance OP as a radius, draw a circle. Can you prove that the sides of triangle ABC are tangent to this circle?

---

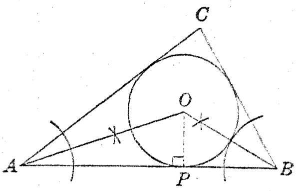
2. Prove that the bisectors of the exterior angles at two vertices and the bisector of the interior angle at the third vertex of any triangle are concurrent.

3. Prove that the radius of a circle inscribed in an equilateral triangle is one-third the altitude, and the radius of the circumscribed circle is two-thirds the altitude.

CONSTRUCTION XVI

274. *To inscribe a circle in a given triangle.*

Given: Triangle ABC.



Required: To inscribe a circle in $\triangle ABC$.

HINT. — Is point O, the intersection of two of the angle bisectors, equidistant from AB, AC, and BC?

275. Incenter. The point of intersection of the bisectors of the angles of a triangle is called the **incenter** of the triangle. It is the center of the inscribed circle.

EXERCISES

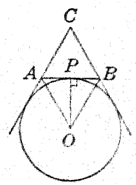
1. Draw an obtuse triangle and inscribe a circle in it.

2. O is the incenter of equilateral triangle ABC. Inscribe a circle in each of the triangles OAB, OAC, and OBC. (Take $AB = 3$ in.)

3. Where do the perpendicular bisectors of the legs of a right triangle intersect? Prove it.

HINT. — See § 153.

4. Prove that the perpendicular bisectors of the four sides of a rectangle are concurrent.



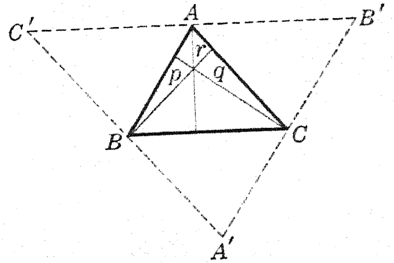
5. In $\triangle ABC$, bisect the exterior angles at A and B. With O, their point of intersection as a center, and a radius equal to $\perp OP$, draw a circle. Prove that this circle is tangent to AB, and to CA and CB produced. It is called an **escribed** circle and its center is called an **excenter** of triangle ABC.

6. Draw an equilateral triangle and construct its inscribed circle and its three escribed circles. (See Ex. 5.)

***7.** Draw a triangle $A'B'C'$. Let A, B, and C be the mid-points of sides $B'C'$, $C'A'$, and $A'B'$, respectively. Show that the perpendicular bisectors of the sides of triangle $A'B'C'$ are the altitudes of triangle ABC.

PROPOSITION 7. THEOREM

276. B. *The altitudes of a triangle are concurrent.*



Given: Triangle ABC with altitudes p, q, and r.

To prove: p, q, and r are concurrent.

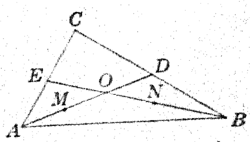
Plan: The sides of $\triangle A'B'C'$ are drawn parallel to the sides of $\triangle ABC$. Prove that the altitudes of $\triangle ABC$ are the perpendicular bisectors of $\triangle A'B'C'$.

277. Orthocenter. The point in which the altitudes of a triangle intersect is called the **orthocenter.**

EXERCISES

1. Why will any two medians of a triangle intersect?

2. If E and D are the mid-points of AC and BC, respectively, ED is equal to half of what segment?



3. If $AM = MO$ and $BN = NO$, can you prove $MN = ED$?

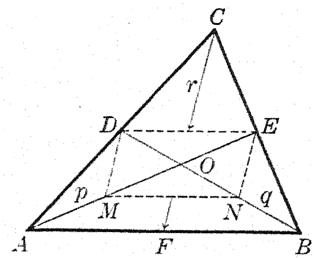
4. Prove $ED \parallel MN$, and hence from Ex. 3 prove that $MNDE$ is a parallelogram.

5. If $MNDE$ is a parallelogram, what can you say about diagonals EN and MD (§ 147)?

6. Since $EO = ON = NB$, and $DO = OM = MA$, how do any two medians of a triangle divide each other? Why can you say the median from C will pass through O?

PROPOSITION 8. THEOREM

278. B. *The medians of a triangle intersect in a point which is two-thirds of the distance from any vertex to the mid-point of the opposite side.*



Given: Triangle ABC, with medians p, q, and r.

To prove: p, q, and r are concurrent in a point O, so that $AO = \frac{2}{3}\,AE$, $BO = \frac{2}{3}\,BD$, and $CO = \frac{2}{3}\,CF$.

Plan: 1. Suppose any two medians AE and BD intersect at O; bisect AO at M and BO at N and prove that $MNED$ is a parallelogram, and that O is the mid-point of ME and ND.

Then $AM = MO = OE$ and $BN = NO = OD$.

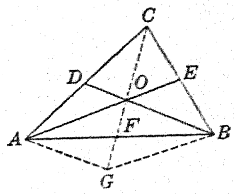
Similarly it can be shown that median CF intersects median AE at O.

Proof: Write the proof in full.

279. Center of gravity of a triangle. The point in which the medians of a triangle intersect is called the **centroid** or **center of gravity** of the triangle.

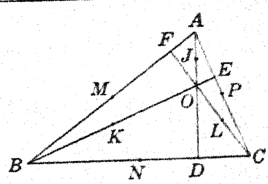
EXERCISES

1. Prove that the median of an equilateral triangle bisects the angle at the vertex, is an altitude, and is the perpendicular bisector of the base.



2. Prove the theorem in § 278 by the following construction: Show that medians AE and BD intersect at a point O. Then draw CF through O, meeting AB at F, and produce it to G, so that $OG = CO$. Draw AG and BG. Show that $AGBO$ is a parallelogram, and hence that $AF = FB$. (See § 147.)

3. If O is the orthocenter of $\triangle ABC$, prove that A is the orthocenter of $\triangle BCO$, B is the orthocenter of $\triangle ACO$, and C is the orthocenter of $\triangle ABO$.



Exercises 4–9 refer to triangle ABC, in which M, N, and P are the mid-points of the sides; AD, BE, and CF are altitudes and O is the orthocenter; J, K, and L are the mid-points of AO, BO, and CO, respectively.

4. Prove that MK and PL are parallel to AO. Similarly show that MJ and NL are parallel to BO; also that NK and PJ are parallel to CO.

HINT. — Recall § 154.

5. Prove that $\angle NMJ$ is a right angle by showing that it is equal to $\angle AEB$.

HINT. — Why is $MN \parallel AC$? See § 116 and Ex. 4.

Also show that $\angle$s PNK and MPL are right angles.

***6.** Prove that $MJLN$ is a rectangle. Also prove that $KNPJ$ and $KLPM$ are rectangles.

HINT. — See Ex. 4–5.

***7.** Prove that the diagonals of the rectangles in Ex. 6 (JN, LM, and KP) are equal and are concurrent in a point H.

***8.** *a.* Prove that a circle with H as center and JN as diameter can be drawn through points M, N, D, and J.

SUGGESTION. — Recall § 260.

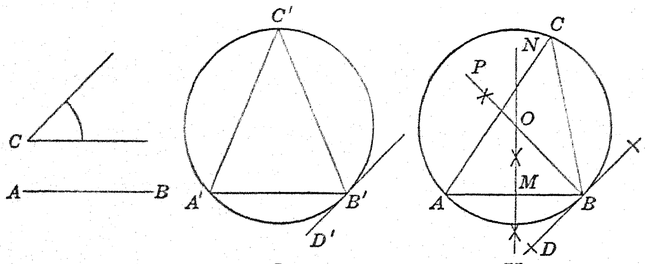
b. In a similar manner show that a circle on KP as diameter can be drawn through points N, E, P, and K.

c. Show that a circle on ML as diameter can be drawn through points P, F, M, and L.

***9.** In Ex. 7 it is shown that JN, ML, and KP are equal and concurrent in a point H. Using the results obtained in Ex. 8, show that a circle with H as center can be drawn through points M, N, P, D, E, F, J, K, and L. This circle is called the *nine-point circle.*

CONSTRUCTION XVII

280. A. *On a given segment as chord construct an arc of a circle in which a given angle may be inscribed.*



I II

Given: Angle C and segment AB.

Required: Construct an arc of a circle on AB as chord, so that every angle inscribed in the arc shall equal angle C.

Analysis: Suppose Fig I represents the completed construction, and that any angle inscribed in $\overset{\frown}{A'C'B'}$ will equal $\angle C$. Since the circle passes through A' and B', its center will be equidistant from A' and B' and will lie on what line (§ 89)?

If $D'B'$ is tangent to the circle at B', how does $\angle A'B'D'$ compare with $\angle C'$? Will a perpendicular to $D'B'$ at B' pass through the center of the circle?

Construction: At B construct $\angle ABD = \angle C$. Then construct MN, the perpendicular bisector of AB. At B construct $BP \perp DB$ and intersecting MN at O (Post. 15).

Construct a circle with center O and radius OA. Then $\overset{\frown}{ACB}$ is the required arc.

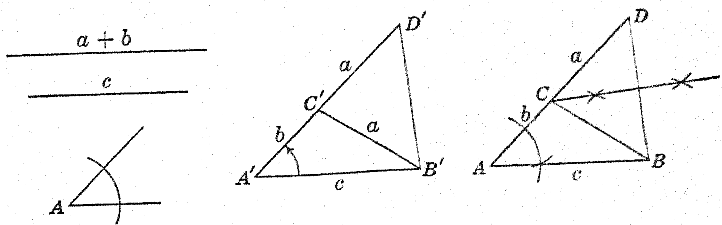
Proof: Write the proof in full.

281. Locus theorem VII. *The locus of the vertex C of a triangle with a given base AB and given angle C is the arc of a circle in which angle C can be inscribed which arc has AB as chord.*

282. A. More difficult constructions. These constructions are more difficult than any that you have had. Some of them will challenge your best efforts. Before doing them review §§ 229–233.

EXERCISES

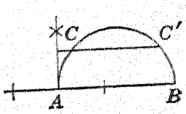
1. Construct a triangle ABC, given $a + b$, c, and $\angle A$.



Plan: $\triangle A'B'C'$ represents the sketch of the completed triangle. Side $A'C'$ is produced to D' so that $A'D' =$ the given length $a + b$. $B'D'$ is drawn.

$\triangle A'B'D'$ can be constructed immediately, since we know two sides and the included angle. Since $\triangle B'C'D'$ is isosceles, C' is equidistant from B' and D'. Hence the perpendicular bisector of BD will locate point C.

2. Construct a right triangle, given an acute angle and the sum of the legs.

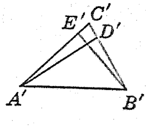


3. Construct a right triangle, given the hypotenuse and the altitude to the hypotenuse.

HINT. — Why are C and C' vertices of the required triangle?

4. Is the construction in Ex. 3 always possible?

5. Construct a triangle, having given a side and the altitudes to the other two sides.



HINT. — On AB as hypotenuse construct right triangles $A'B'D'$ and $A'B'E'$.

6. Construct a triangle, having given a side, the altitude to the given side, and the radius of the circumscribed circle.

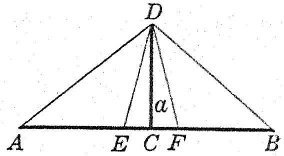
HINT. — Draw the circle first and then draw the given side a chord in the circle.

7. Construct a parallelogram, given a diagonal and two sides.

***8.** Construct a triangle, having given the medians.

Recall § 278 and see Ex. 2, § 279, using the figure of this exercise for analysis and discuss how to construct the triangle.

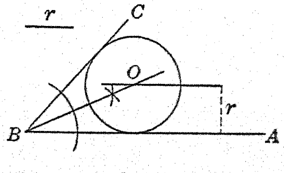
9. Construct an isosceles triangle, having given the perimeter and the altitude to the base.



ANALYSIS. — If a is the given altitude, AB the given perimeter, and DEF the required triangle, then $DE = AE$ and $DF = BF$. How, therefore, are E and F located?

Make the constructions and write the construction and proof in full.

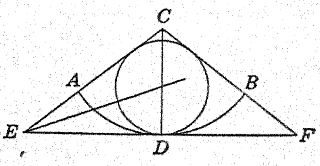
10. Construct a circle with a given radius and tangent to the sides of a given angle.



ANALYSIS. — If r is the given radius and $\angle ABC$ the given angle, the center O must be on a line parallel to AB, at the distance r from AB, and also on the bisector of $\angle ABC$, and hence is their intersection.

Make the construction, and write the construction and proof in full.

11. Construct a circle tangent to a given arc and to the sides of the central angle which intercepts the arc.



ANALYSIS. — To what point of $\widehat{AB}$ is radius CD drawn? How is EF drawn? How are points E and F located? Then the circle is inscribed in $\triangle EFC$.

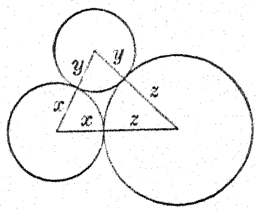
12. Construct within a given square four equal circles each tangent to two others and to two sides of the square.

13. Construct a circle with given radius tangent to a given circle and to a given straight line.

HINT. — If a circle with radius r be tangent to circle O, what is the locus of its center? What is the locus of the center of a circle with radius r tangent to a line l? Find the intersection of these loci.

14. Construct a circle with given radius which shall be tangent to two given circles.

***15.** Construct three circles each tangent to the other two, with the vertices of a given triangle as centers.



ANALYSIS. — If x, y, and z are the required radii,

$$x + y = b \quad (1)$$
$$x + z = c \quad (2)$$
$$y + z = a \quad (3)$$

Solve for x, y, and z. See *Review of Algebra* for solution.

Make the construction, and write the complete construction and proof.

***16.** Construct a circle which shall pass through a given point and be tangent to a given circle at a given point.

***17.** Construct a circle with given radius which shall pass through a given point and be tangent to a given circle.

***18.** Through a given point draw a straight line cutting a given circle so that the chord intercepted on it by the circle shall equal a given length.

***19.** Construct a circle which shall be tangent to a given circle and to two equal chords of the given circle.

***20.** Through an intersection of two given circles draw a segment of given length terminating in the two circles.

HINT. — If O and O' are the centers of the circles and A is a point of intersection, on OO' as hypotenuse construct a right triangle with one leg $O'C$ equal to half the given segment. Through A draw a line $\parallel O'C$, and terminated by the circles.

*21. Construct a chord of given length in a given circle which, if extended, intersects a given line outside the circle at right angles.

*22. Construct a chord of given length in a given circle parallel to a given line outside the circle.

*23. Construct a chord in a given circle at a given distance from the center which, if extended, makes a given angle with a given line.

*24. Prove that the circumcenter of a triangle is the orthocenter of another triangle formed by joining the mid-points of the sides of the first.

*25. The circle through the mid-points of the sides of a triangle passes through the feet of the altitudes of the triangle.

283. Summary of the Work of Unit Five. In this unit you have learned about:

I. ***The measurement of angles in circles by means of their arcs.***

1. *A central angle is measured by its intercepted arc.*
2. *An inscribed angle is measured by half the intercepted arc.*
3. *An angle formed by a tangent and a chord drawn to the point of contact is measured by half the intercepted arc.*
4. *An angle formed by two chords intersecting within a circle is measured by half the sum of the intercepted arcs.*
5. *An angle formed by two secants, by a secant and a tangent, or by two tangents, intersecting outside the circle is measured by half the difference of the intercepted arcs.*

II. ***The locus of points which satisfy some given conditions.***

In solving a locus problem you have learned:

1. *Get a general idea of the form of the locus by locating a sufficient number of points.*
2. *Accurately define what the locus is.*
3. *Prove the correctness of your statement in* (2).

   This is done when you prove:

   a. *Every point on the locus satisfies the given conditions;* and either

   b. *Every point satisfying the given conditions lies on the locus;* or

   c. *A point which does not satisfy the requirement does not lie on the locus.*

III. ***The fundamental locus theorems.***

1. *The locus of points at a given distance from a given point is a circle with the given point as center and with the given distance as radius.*
2. *The locus of points each equidistant from two given points is the perpendicular bisector of the segment connecting the points.*
3. a. *The locus of points equidistant from the sides of an angle is the bisector of the angle;* or

   b. *The locus of points equidistant from two intersecting straight lines is the pair of lines bisecting the angles formed.*
4. *The locus of points equidistant from two parallel lines is the line parallel to each of them and midway between them.*

5. *The locus of points at a given distance from a given line is a pair of lines, one on either side of the given line, each parallel to the given line, and at the given distance from it.*

6. *The locus of the vertex of the right angle of a right triangle having a given hypotenuse is a circle having the given hypotenuse as diameter.*

7. *The locus of the vertex C of a triangle with a given base AB and given angle C is the arc of a circle in which $\angle C$ can be inscribed and which arc has AB as chord.*

IV. Concurrent lines in a triangle.

1. *The perpendicular bisectors of the sides are concurrent at the center of the circumscribed circle (circumcenter).*

2. *The bisectors of the angles are concurrent at the center of the inscribed circle (incenter).*

3. *The altitudes of a triangle are concurrent (orthocenter).*

4. *The medians of a triangle are concurrent (centroid, center of gravity).*

V. Constructions.

1. *Construct a tangent to a circle from a given outside point.*

2. *Inscribe a circle in a triangle.*

3. *Construct, on a given segment as chord, an arc of a circle in which a given angle can be inscribed.*

REVIEW OF UNIT FIVE

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the test. The references given are those most closely related to the exercise.

1. State any additional ways you have learned of proving angles equal; of proving arcs equal. §§ 241–248.

2. Give all the theorems about the measurement of angles which intercept arcs on a circle. §§ 237, 241–248.

3. What do we mean when we say a central angle has the same measure as its intercepted arc? § 237.

4. Distinguish between an angle degree and an arc degree. §§ 235, 236.

5. Do the incenter, orthocenter, and circumcenter of a triangle ever coincide? §§ 271, 275, 277.

6. How many arcs does a chord have? § 187.

7. Can two lines be said to be concurrent? § 268.

8. Is the point of intersection of the bisectors of the angles of a triangle ABC equidistant from A, B, and C? § 256.

9. How many chords may be drawn from a point on a circle? How many diameters? How many tangents?

10. If two circles are unequal, can they have arcs containing the same number of arc degrees? Can the arcs be made to coincide? § 236.

11. If two unequal circles have equal central angles, how do the intercepted arcs compare in arc degrees? Do you think they will have the same length? § 236.

Complete the following statements:

12. An inscribed angle . . . . . § 241.

13. An angle inscribed in a semicircle . . . . . § 243.

14. An angle formed by two secants . . . . . § 248.

15. An angle formed by two tangents . . . . . § 248.

16. An angle formed by a tangent and a secant ..... § 248.

17. An angle formed by a tangent and a chord ..... § 245.

18. An angle formed by two intersecting chords ..... § 246.

19. The perpendicular bisectors of the sides of a triangle ..... § 270.

20. The bisectors of the angles of a triangle ..... § 273.

21. The altitudes of a triangle ..... § 276.

22. The medians of a triangle ..... § 278.

23. What is the orthocenter of a triangle? The center of gravity? §§ 277, 279.

24. What is another name for the center of gravity of a triangle? § 279.

25. In what kind of a triangle do the centroid, incenter, circumcenter, and orthocenter coincide?

26. What are the steps in proving a locus theorem? § 251.

27. What are the steps in making a construction? § 232.

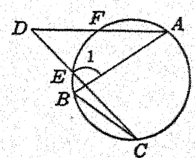
28. State the seven fundamental locus theorems. § 283 – III.

NUMERICAL EXERCISES

1. A central angle of a circle is 35° and its chord is 3 in. long. How does this chord compare with the chord of a central angle of 70°? Show that the second chord is less than twice the first.

2. The diameter of a circle is 6 in. What is the length of the radius? Of the longest chord in the circle?

3. How many tangents can be drawn to a circle from a point on the circle? From an outside point? From an inside point?

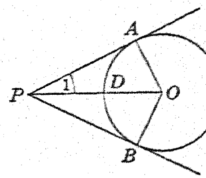


4. $\widehat{AC}$ is 140°. How many degrees in $\angle B$?

5. If $\angle 1$ is 95° and $\widehat{BC}$ is 100°, how many degrees in $\widehat{EA}$?

6. If $\angle D$ is 40° and $\widehat{EF}$ is 40°, how many degrees in $\widehat{AC}$?

7. Radii OA and OB are perpendicular to PA and PB, respectively. If PA is 8 in., how long is PB?



8. If $\angle 1$ is 30°, how many degrees in $\overset{\frown}{DB}$?

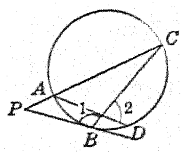
9. If $\overset{\frown}{AB}$ is 110°, how large is $\angle P$?

10. If $\overset{\frown}{AEB}$ is 245°, how large is $\angle O$?

11. If $\overset{\frown}{AB}$ is 100°, how large is $\angle PBA$?

12. If $\overset{\frown}{AB}$ is 55°, AC is a diameter, and $\overset{\frown}{BD}$ is 40°

a. How many degrees in $\angle P$?
b. How many degrees in $\angle 2$?
c. How many degrees in $\angle 1$?
d. Is $AD \parallel PB$?

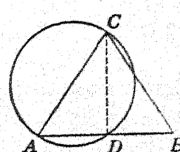


GENERAL EXERCISES

1. The bisector of an inscribed angle bisects the intercepted arc.

2. The bisectors of all angles inscribed in a given arc of a circle are concurrent.

3. If a circle is drawn with one of the equal sides of an isosceles triangle as its diameter, it bisects the base.



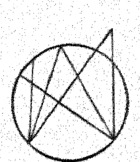
4. A parallelogram inscribed in a circle is a rectangle.

5. If $ABCD$ is an inscribed square and M any point on $\overset{\frown}{AB}$, then MC and MD trisect (divide into three equal parts) $\angle AMB$.

6. Chords AB and CD intersect within a circle at M. Prove that $\triangle BCM$ and $\triangle ADM$ are mutually equiangular, that is, the angles of one, respectively, equal the angles of the other.

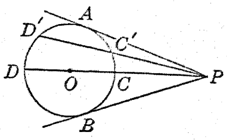
7. Chords AB and CD extended meet outside of a circle at M. Prove $\triangle BCM$ and $\triangle ADM$ mutually equiangular.

8. If an angle is greater than the inscribed angle of a circle which intercepts the same arc, its vertex is within the circle; if it is equal to the inscribed angle, its vertex is on the circle; and if it is less than the inscribed angle, its vertex is outside of the circle.



SUGGESTION. — In each case find the measure of the angle.

9. Draw a large figure like the one shown. Make the radius of the circle about 2 in. and the distance PO about 4 in. From P draw the tangents PA and PB, the secant PCD through O, and a number of other secants like $PC'D'$. Mark the mid-points of PA, PB, of each secant PD' and of each external segment PC'. What does the locus of the mid-points seem to be?



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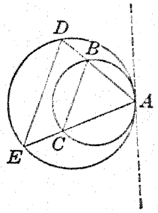
10. Two circles are tangent internally at A. Any two chords ABD and ACE are drawn, cutting the circles at B, D, C, and E, as in the figure. Prove $DE \parallel BC$.



11. Is the statement in Ex. 10 true when the circles are externally tangent? Prove it.

12. If two circles intersect and a straight line is drawn parallel to their common chord and cutting both circles, the segments of the line intercepted between the circles are equal.

13. A circle is drawn with the radius of another circle as its diameter. Through A, their point of contact, any chord AB of the larger circle is drawn, intersecting the smaller circle at C. Prove that C bisects AB.

14. Show that the locus of the centers of all circles tangent to a given circle at a given point is the straight line determined by the given point and the center of the given circle.

HINT. — What two things must be proved (§ 251)?

15. What is the locus of the centers of all circles with a given radius r and tangent externally or internally to a given circle? Prove the answer.

16. If two circles are externally tangent and a line is drawn through the point of contact intersecting the circles, tangents to the circles at the points of intersection are parallel.

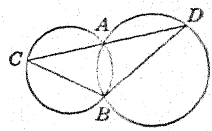
SUGGESTION. — Draw the common internal tangent. What is the relation between the angles formed?

17. What is the locus of the vertices of triangles having a given base AB and a vertex angle of 45°? What is the locus of the vertices of triangles having a given base AB and a given vertex angle? Prove your answer.

18. Construct a right triangle, given the hypotenuse and the sum of the legs.

19. Prove that the locus of the middle points of all of the equal chords of a circle is a concentric circle.

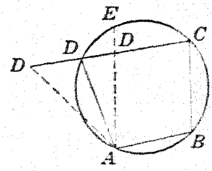
***20.** Prove that the locus of the mid-points in Ex. 9 is a circle with radius $\frac{1}{2}$ OC and with center O' at the mid-point of MM', where M and M' are the mid-points of PD and PC, respectively.



***21.** Two circles intersect at A and B, and CD is any line segment drawn through A terminating in the circles. Prove that $\angle DBC$ is constant (has the same size) for all positions of CD.

Hint. — $\angle$s C and D have the same measure as what arcs?

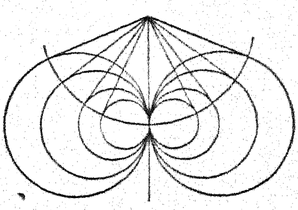
***22.** Through A, a point of intersection of two equal circles, two line segments BC and DE are drawn, terminating in the circles. Prove the chords BD and EC equal.



***23.** If the opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about it.

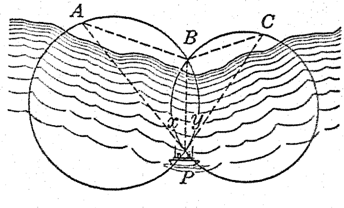
Suggestion. — Draw a circle passing through A, B, and C, and prove by the indirect method that it passes through D. There are two possibilities: D is either on the circle or it is not. If it is not on the circle we have (1) D is outside the circle, or (2) D is inside the circle. Show that in (1) and (2) $\angle D$ is not supplementary to $\angle B$.

***24.** A circle can be circumscribed about an isosceles trapezoid.



***25.** Circles are drawn tangent to a given line at the same point of it. From another point of the line tangents are drawn to all of the circles. Prove that the locus of the points of contact of these tangents is a circle.

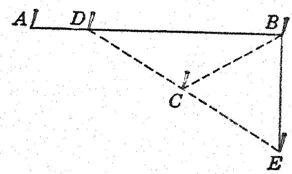
***26.** An important problem in marine surveying is to determine the position P of a boat from which soundings are being made along a coast. The boat moves from place to place, and it is necessary to locate these positions on the chart. Three stations, A, B, and C, are located on the shore. Angles x and y are observed from the boat. A, B, and C are located on the chart. P is located on the chart by the intersection of two circles, one passing through A and B, and the other through B and C. Show how to locate their centers. Suppose that $\angle x = 40°$ and $\angle y = 70°$. (See § 280.)



PRACTICAL APPLICATIONS

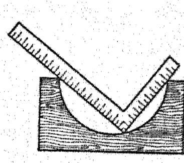
(OPTIONAL)

1. Prove that the following method may be used by a surveyor to lay out a line perpendicular to AB at B: Select any point C 50 ft. from B. With one end of a 50-ft. tape at C, swing the other end to D, in line with A and B. With one end of the tape still at C, swing the other end to locate E in line with D and C. Then BE is the required perpendicular.

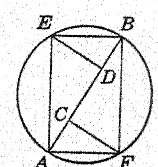


SUGGESTION. — A circle may be drawn through D, B, and E.

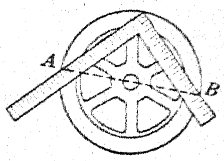
2. Pattern makers and others use the carpenter's square as follows to determine if a half round hole is a true semicircle: The square is placed as in the figure. If the heel of the square touches the bottom of the hole in all positions of the square, while the blades rest against the edges of the hole, the hole is a true semicircle. Prove that the method is accurate.



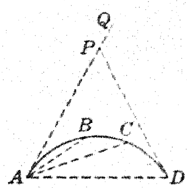
3. The strongest beam that can be cut from a round log has a cross section which may be constructed as follows: Draw diameter AB and trisect it at C and D. Draw $DE \perp AB$ and $CF \perp AB$. Draw AE, EB, BF, and FA. Prove that $AEBF$ is a rectangle.



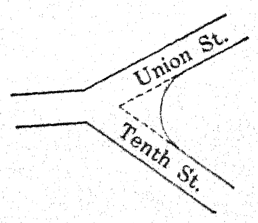
4. Prove that the center of any circular object may be located by means of a carpenter's square by the following method: Lay the square on the object, with the heel at the rim, and mark the points A and B where the blades cross the rim. Now, by placing the blade of the square on A and B, find the middle point of AB. That is the center.



5. In railroad surveying, curves are laid out by turning off equal angles and setting stakes every 100 ft. If the curve begins at A, $\angle BAP$ is turned off from the tangent AP and AB measured 100 ft., then $\angle CAB$ is turned off and BC measured 100 ft., etc., the process being continued until the curve ends in the tangent DP at D. Since the curve is to be an arc of a circle, prove that $\angle BAP$, $\angle CAB$, etc., must be made equal, and that each must equal one half the central angle of a 100-ft. chord.

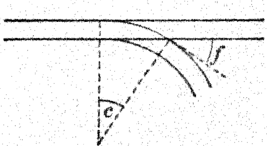


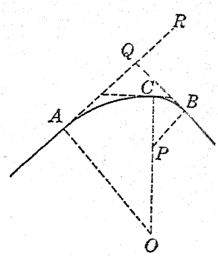
6. In Ex. 5, $\angle DPQ$ is called the *intersection* angle, and $\angle BAP$, $\angle CAB$, etc., the *deflection* angles of the curve. Prove that the intersection angle equals twice the sum of all the deflection angles at A. Show that the intersection angle has the same measure as the central angle of the curve.



7. Draw the *connecting curve* of given radius r at the intersection of the curbs of two given streets.

8. In laying out a switch on a railroad track, a "frog" is used at the intersection of the two rails, to allow the flanges of wheels rolling on one rail to cross the other. If one track is straight and the other one curved, prove that the angle f of the frog equals the central angle c of the curve. (Note that $\angle f$ is formed by the straight rail and the tangent to the curved rail at the point where the two rails cross.)





9. The railroad curve ACB is a compound curve, composed of $\widehat{AC}$ and $\widehat{CB}$, with centers at O and P, respectively. Point P is on OC. Prove that the intersection angle BQR equals the sum of the central angles, $\angle COA$ and $\angle BPC$.

SUGGESTION. — Draw the common tangent at C.

PRACTICE TESTS

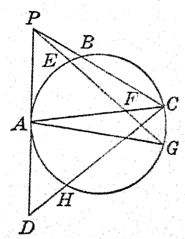
These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT FIVE

TEST ONE

Numerical Exercises

In Ex. 1–5 PD is tangent to the circle at A and PC, AC, and DC intersect at C.



For Ex. 1–8, see §§ 241–248.

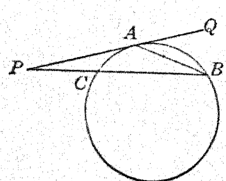
1. If $\widehat{AB} = 70°$, how large is $\angle BCA$?

2. If $\widehat{EBC} = 90°$ and $\widehat{AHG} = 170°$, how large is $\angle CFE$?

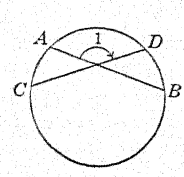
3. If $\widehat{AH} = 64°$ and $\widehat{ABC} = 150°$, how large is $\angle D$?

4. If $\widehat{AEG} = 190°$, how large is $\angle DAG$?

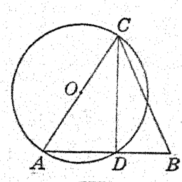
5. If $\widehat{BE} = 10°$ and $\widehat{CG} = 40°$, how large is $\angle GPC$?



I



II



III

6. (Fig. I) $\widehat{AC}$ is 50°, PA is a tangent, and $\angle P$ is 7°, find $\angle BAQ$.

7. (Fig. II.) If $\widehat{AC}$ is 30° and $\widehat{DB}$ is 50°, how large is $\angle 1$?

8. (Fig. III) AC is a diameter of circle O and $\triangle ABC$ is isosceles ($AC = BC$). AB intersects the circle at D. If $\angle B = 50°$, how large is $\widehat{AD}$?

9. A chord of a circle is equal to the radius. How many degrees in the major arc of the chord? § 238, Ex. 3.

10. How many lines can you draw, all points of which are 4 in. from an indefinitely extended line AB? § 259.

11. What is the greatest possible number of points in the locus of points 2 in. from a given point P and also 1 in. from a given line l. §§ 252, 259.

12. The base of a triangle is the fixed segment AB and its altitude is 5 in. How many such triangles can you draw? § 259.

TEST TWO

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. An angle made by two secants drawn to a circle from an external point is measured by half the *sum* of the intercepted arcs. § 248.

2. A circle may be inscribed in an *obtuse* triangle. § 274.

3. The locus of the vertex of a right triangle with a given hypotenuse is a circle with the hypotenuse as its *diameter*. § 260.

4. The *altitudes* of a triangle are concurrent in a point two-thirds the distance from one vertex to the mid-point of the opposite side. § 278.

5. The locus of a point 3 inches from a given line is *one* straight line. § 252.

6. The angle between two tangents intersecting outside the circle is measured by half the *sum* of the intercepted arcs. § 248.

7. An angle inscribed in an arc *greater* than a semicircle is an acute angle. § 241.

8. There may be not more than *four* points equidistant from two given points A and B and at the same time 3 inches from a given point P. §§ 252, 254.

9. The point of intersection of the angle bisectors of a triangle is the center of the *inscribed* circle. § 275.

10. The point of intersection of the *altitudes* of a triangle is called the orthocenter of the triangle. § 277.

11. A circle can be circumscribed about a quadrilateral if, and only if, the opposite angles are *supplementary*. Page 267, Ex. 23.

12. The locus of the mid-points of parallel chords of a circle is the diameter *parallel* to the chords. § 195.

TEST THREE

Matching Exercises

Under B a brief description is given or a comment is made about the partial statements under A. Match them correctly.

A	B
I. An angle inscribed in a semicircle. § 243.	1. Is measured by half the sum of the intercepted arcs.
II. Complementary arcs. § 238.	2. Is a pair of lines bisecting the angles formed.
III. An arc degree. § 236.	3. Is a right angle.
IV. An angle formed by two tangents. § 248.	4. Are three or more lines which have a point in common.
V. An angle formed by two intersecting chords. § 246.	5. Is called the circumcenter.
VI. An angle formed by a tangent and a chord drawn to the point of contact. § 245.	6. Is a circle.
VII. The point of intersection of the angle bisectors of a triangle. § 275.	7. Is an arc which is one ninetieth of a quadrant.
VIII. The point of intersection of the perpendicular bisectors of the sides of a triangle. § 271.	8. Are two arcs whose sum is a quadrant.

IX. The locus of points equidistant from two intersecting lines. § 257.

X. The locus of points at a given distance from a given line. § 259.

XI. The locus of points at a distance d from a given point P. § 252.

XII. Concurrent lines. § 268.

9. Is measured by half the difference of the intercepted arcs.

10. Is measured by one half the intercepted arc.

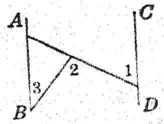
11. Is a pair of lines parallel to the given line and at the given distance from it.

12. Is called the incenter.

CUMULATIVE TESTS ON THE FIRST FIVE UNITS

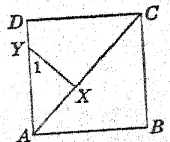
TEST FOUR

Numerical Exercises



1. What is the sum of the angles of a pentagon? § 133.

2. AB is parallel to CD, $\angle 1 = 70°$, $\angle 3 = 30°$. How many degrees in $\angle 2$? §§ 113, 128.



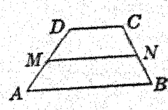
3. $ABCD$ is a square. If XY is perpendicular to AC, how many degrees in $\angle 1$? §§ 142, 125.

4. If one acute angle of a right triangle is four times the other, how many degrees in each? § 125.

5. One diagonal of a rhombus is equal to a side. How many degrees in each angle of the rhombus? § 71.

6. Angle A is formed by two tangents touching a circle at B and C. If $\widehat{BC}$ is 120°, how large is $\angle A$? § 248.

7. In a circle a chord is equal to the radius. How many degrees in its arc? § 238, Ex. 3.



8. In trapezoid $ABCD$, $AB \parallel CD$. M and N are mid-points of AD and BC respectively. If DC is 5 in., and MN is 8 in., find AB. § 157.

9. Triangle ABC has a right angle at C. If $AB = 16$ in., how long is median CM? § 159.

10. In triangle ABC, $AB = 8$ in., and $AC = 6$ in. Side BC must be greater than what value? § 175, Ex. 9, 10.

11. In Ex. 9, if $AC = 8$ in., how large is $\angle A$? § 161.

12. A parallelogram has a fixed base AB and an altitude of 2 in. How many such parallelograms can you draw?

TEST FIVE

True-False Statements

If a statement is always true, mark it so. If not, replace each word in italics by a word which will make it a true statement.

1. The diagonals of a parallelogram bisect *the angles*. § 147.

2. The diagonals of a *rhombus* are perpendicular to each other. § 147, Ex. 1.

3. The opposite angles of a parallelogram are *supplementary*. § 141.

4. If the sum of the interior angles of a polygon equals the sum of the exterior angles, the polygon is a *quadrilateral*. §§ 133, 134.

5. The hypotenuse of a right triangle equals twice the *median* drawn to the hypotenuse. § 159.

6. If the sum of two angles of a triangle equals the third angle, the triangle is a *right* triangle. § 125.

7. If a statement is true, its converse is *always* true. § 74.

8. If two angles have their sides parallel, they are either equal or *complementary*. § 116.

9. The diagonals of a *rhombus* are equal. § 162, Ex. 6.

10. The angle opposite the longest side of a triangle may be *acute*. § 172.

11. If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are *complementary*. § 115.

12. The line which passes through the mid-points of two *equal* chords passes through the center of the circle. § 197.

TEST SIX

Multiple-Choice Statements

From the expressions printed in italics select that one which bests completes the statement.

1. Complementary angles are *angles having 45°, adjacent angles, angles whose sum is 90°*. § 38.

2. The locus of points at a given distance from a given point is *the center of a circle, a circle with its center at the given point, a perpendicular bisector*. § 252.

3. The number of triangles that can be constructed on a fixed base AB with an altitude of 4 in., is *one, two, as many as you please*. § 259.

4. An acute triangle is a triangle having *one, two, three*, acute angles. § 30.

5. The sum of the angles of a hexagon is *360°, 600°, 720°*. § 133.

6. The circumcenter of a triangle is the point of intersection of the *medians, angle bisectors, perpendicular bisectors of the sides*. § 271.

7. The converse of a theorem is *sometimes, always, never* true. § 74.

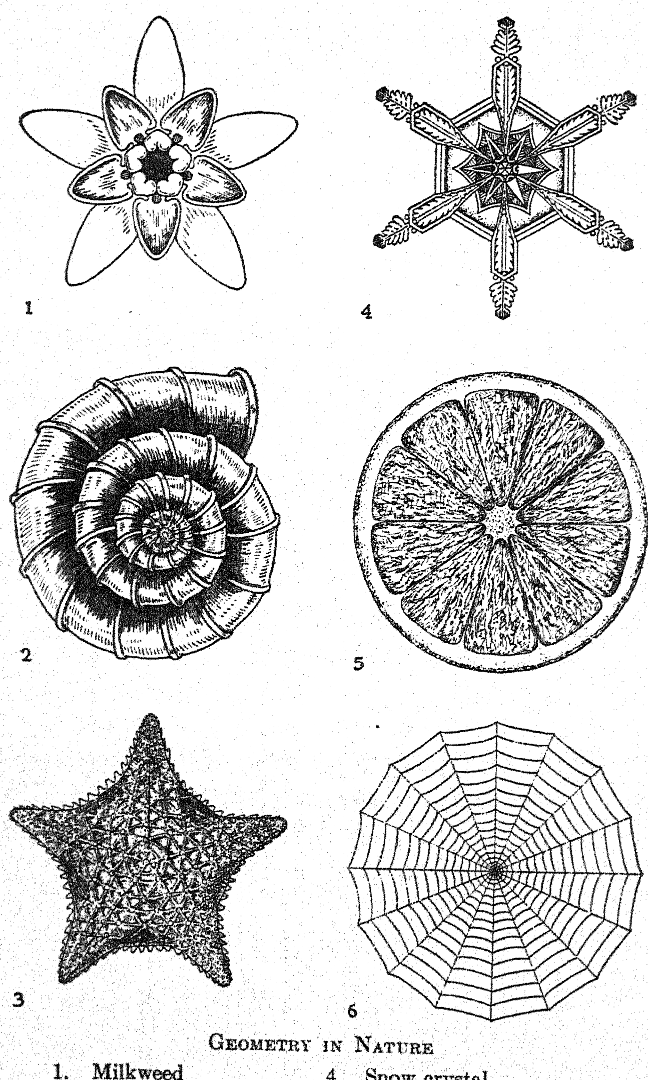
8. The indirect method of proof is frequently used to prove *axioms, the converse of a theorem, corollaries*. § 101.

9. A triangle cannot be constructed unless the sum of two given sides is *equal to, less than, greater than* the third side. § 175, Ex. 9.

10. Each angle of a regular twelve-sided polygon contains *150°, 135°, 120°*. § 133.

11. The sum of the exterior angles of any polygon is *180°, 360°, you cannot tell*. § 134.

12. The point of intersection of the medians of a triangle is *the center of the inscribed circle, two-thirds the distance from a vertex to the mid-point of the opposite side, the center of the circumscribed circle*. § 278.



GEOMETRY IN NATURE

1. Milkweed
2. A mollusk
3. Starfish
4. Snow crystal
5. Cross-section of orange
6. Spider web

UNIT SIX

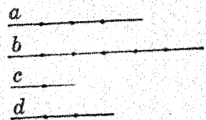
SIMILAR POLYGONS (BOOK THREE)

Proportion and Similar Triangles

284. Similar figures. When you look through a magnifying glass, you see an image larger than the object at which you are looking, but having *the same shape*. Figures related as are the object and image are called **similar figures.** In order to study similar geometric figures, however, we shall need a definition worded in mathematical terms. We shall denote the amount of magnification of the figure by means of a **ratio.**

285. The meaning of ratio. The ratio of two quantities is the quotient of their numerical measures *in terms of a common unit.*

Thus the ratio of $1\frac{1}{2}$ in. to $7\frac{1}{2}$ in. is $1\frac{1}{2} \div 7\frac{1}{2}$, or $\frac{1}{5}$. The ratio of 1 pint to $3\frac{1}{2}$ gallons is $\frac{1}{28}$, since there are 28 pints in $3\frac{1}{2}$ gallons.



In the figure at the right a common unit has been applied to segments a, b, c, and d. The ratio of a to b is $\frac{4}{6}$; of a to c is $\frac{4}{2}$; of c to d is $\frac{2}{3}$. We write the ratio of a to b as $\dfrac{a}{b}$ or, with two dots, $a : b$.

286. Proportion. An equality between two equal ratios is called a **proportion.**

Thus in line segments a, b, c, and d, the ratio of a to b

is $\frac{4}{6}$; the ratio of c to d is $\frac{2}{3}$. Since $\frac{4}{6} = \frac{2}{3}$, we say that the *four* quantities a, b, c, and d are in proportion. That is, $\frac{a}{b} = \frac{c}{d}$.

287. Terms of a proportion. The terms a, b, c, and d are called the *first, second, third,* and *fourth* terms, respectively, of the proportion.

The first and fourth terms are called the **extremes** and the second and third terms, the **means** of the proportion.

We may read the proportion $\frac{a}{b} = \frac{c}{d}$ in one of two ways; thus, "the ratio of a to b is the same as the ratio of c to d"; or, "a is to b as c is to d." Sometimes the proportion is written $a : b = c : d$.

EXERCISES

NOTE. — Refer to the review of algebra in the *Appendix.*

1. What is the ratio of $3\frac{1}{2}$ in. to 1 yd.?

2. What is the ratio of 1 ft. to 1 in.?

3. What is the ratio of $\frac{1}{2}$ right angle to 3 straight angles?

State the extremes and the means in each of the following proportions:

4. $\frac{3}{8} = \frac{10}{a}$ **5.** $\frac{6}{x} = \frac{5}{9}$ **6.** $\frac{12}{20} = \frac{y}{5}$ **7.** $\frac{t}{4} = \frac{7}{8}$

8–11. Solve for the literal number in each proportion in Ex. 4–7.

Express each of the following ratios in simplest form:

12. $\frac{2-5}{15}$ **13.** $\frac{m^2 - n^2}{m - n}$ **14.** $\frac{3\frac{1}{4}}{2\frac{1}{2}}$ **15.** $\frac{2\frac{1}{3}}{7}$ **16.** $\frac{.03}{3}$

Find the value of x in each proportion:

17. $\frac{a}{b} = \frac{c}{x}$ **18.** $\frac{v}{x} = \frac{1}{v}$ **19.** $\frac{x}{a+n} = \frac{a-n}{b}$ **20.** $\frac{x+2}{x+3} = \frac{4}{7}$

Find the ratio of the literal numbers if:

21. $3m = 4n$ **22.** $4y = 8x$ **23.** $x = 3y$ **24.** $a = 1\frac{1}{2}b$

Find the ratio of a to b if:

25. $am = bn$ **26.** $\dfrac{a+b}{a-b} = \dfrac{3}{5}$ **27.** $ay = b(x+y)$

28. $\dfrac{2\pi r}{a} = \dfrac{2\pi R}{b}$ **29.** $3a - 6b = 0$ **30.** $\dfrac{1}{a} = \dfrac{3}{a+b}$

31. A segment 18 in. long is divided into two parts whose ratio is 2 to 7. Find the length of each part.

32. Two complementary angles are in the ratio of 1 to 5. How large is each?

33. The acute angles of a right triangle are in the ratio 2 to 13. How many degrees in each?

34. If the ratio of two supplementary angles is 3 to 7, how many degrees in each?

35. The angles of a triangle are in the ratio of 1 to 2 to 3. How large is each?

36. If $\dfrac{a}{b} = \dfrac{c}{d}$, why is $\dfrac{ma}{mb} = \dfrac{c}{d}$?

288. The formula for a proportion. The formula $\dfrac{a}{b} = \dfrac{c}{d}$ can be considered a general statement of a proportion. By writing this in different forms as in the exercises below, we shall discover some interesting theorems.

EXERCISES

1. If $\dfrac{a}{b} = \dfrac{c}{d}$, then $ad = bc$. To clear of fractions multiply both sides by ——. What axiom is used?

2. From the result in Ex. 1 complete: If four numbers are in proportion, then the product of the —— is equal to the product of the ——. (What are the first and fourth terms called? The second and third?)

3. If $\dfrac{a}{b} = \dfrac{c}{d}$, we know from Ex. 1 that $ad = bc$. What do you get if both sides of this second equation are divided by cd? Prove that if $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a}{c} = \dfrac{b}{d}$.

4. Complete from the result in Ex. 3: If four numbers are in proportion, the —— term is to the third as the —— is to the fourth.

5. By what must you divide both sides of $ad = bc$ to get $\frac{b}{a} = \frac{d}{c}$? From this result complete the statement: If four terms are in proportion, the second term is to the —— as the —— is to the ——.

6. If $\frac{a}{b} = \frac{c}{d}$ and $a = c$, what can you say about b and d? Why? Complete: In a proportion, if the numerators are equal the —— are equal.

NOTE. — The numerators are sometimes called **antecedents** and the denominators are called **consequents**.

7. If $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{b} = \frac{c}{x}$, what conclusion can you draw about d and x? Can you state the result as a theorem?

8. See how many different proportions you can make by using the four numbers in the equation $3 \times 8 = 4 \times 6$. You should get four. In how many of the proportions do you use 3 and 8 as means? 3 and 8 as extremes?

9. If $ad = bc$, write two proportions using b and c as means and two using b and c as extremes.

10. If you add 1 to each side of the equation $\frac{3}{4} = \frac{6}{8}$, do you get $\frac{3+1}{4} = \frac{6+1}{8}$?

11. Add 1 to each side of the equation $\frac{a}{b} = \frac{c}{d}$ and simplify. Do you get $\frac{a+b}{b} = \frac{c+d}{d}$?

12. Can you prove $\frac{a+b}{a} = \frac{c+d}{c}$? What axioms do you use? (Begin by using the result obtained in Ex. 5. Then add 1 to both sides.)

13. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a-b}{b} = \frac{c-d}{d}$. Give a numerical example.

14. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a-b}{a} = \frac{c-d}{c}$.

289. Fundamental theorems. The results obtained in § 288 are used so often we shall express them as theorems.

1. *In any proportion, the product of the means equals the product of the extremes.* (Ex. 1, 2)

2. *In any proportion, the first term is to the third as the second term is to the fourth.* (Alternation) (Ex. 3, 4)

3. *In any proportion, the second term is to the first as the fourth is to the third.* (Inversion) (Ex. 5)

4. *If the two numerators of a proportion are equal, the denominators are equal.* (Ex. 6)

5. *If three terms of one proportion are equal, respectively, to three terms of another proportion, the remaining terms are equal.* (Ex. 7)

6. *If the product of two numbers is equal to the product of two other numbers, either two may be made the means in a proportion in which the other two are the extremes.* (Ex. 8, 9)

7. *In a proportion, the sum of the first two terms is to the second (first) as the sum of the last two is to the fourth (third).* (Addition) (Ex. 11, 12)

8. *In a proportion, the difference between the first two terms is to the second (first) as the difference of the last two is to the fourth (third).* (Subtraction) (Ex. 13, 14)

EXERCISES

1. Write the proportion $\frac{3}{8} = \frac{9}{24}$ by alternation; by inversion; by addition; by subtraction.

2. Write four proportions involving the four numbers in each of the following equations.

(*a*) $4 \times 3 = 2 \times 6$; (*b*) $am = bx$; (*c*) $1 \cdot x = bc$; (*d*) $x = \frac{a}{b}$.

3. Write the proportion $\frac{a}{b} = \frac{c}{d}$ by subtraction; and then write the resulting proportion by alternation.

4. Write the proportion $\frac{a}{b} = \frac{c}{d}$ by alternation, and then write the resulting proportion by addition.

5. From $\frac{a-b}{b} = \frac{c-d}{d}$ and $\frac{a+b}{b} = \frac{c+d}{d}$, how can you obtain $\frac{a-b}{a+b} = \frac{c-d}{c+d}$?

Given the proportion $\frac{a}{b} = \frac{c}{d}$, tell how each of the following statements is obtained.

6. $\frac{b}{a} = \frac{d}{c}$ 7. $\frac{a}{c} = \frac{b}{d}$ 8. $\frac{1}{ad} = \frac{1}{bc}$ 9. $\frac{a+b}{c+d} = \frac{a}{c}$ 10. $\frac{xa}{yb} = \frac{xc}{yd}$

11. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

12. If $\frac{a}{b} = \frac{c}{d}$, is $\frac{a+1}{b+1} = \frac{c+1}{d+1}$?

13. If $\frac{a}{b} = \frac{c}{d}$, is $\frac{a}{d} = \frac{b}{c}$?

14. Transform the proportion $a : b = c : x$ so that x is the first term; the second term; the third term.

15. Construct a triangle with $a = 1\frac{1}{2}$ in., $b = 2$ in., and $c = 2\frac{1}{4}$ in. Construct another triangle with $a' = 3$ in., $b' = 4$ in., and $c' = 4\frac{1}{2}$ in. What is the ratio of $\frac{a}{a'}$? $\frac{b}{b'}$? $\frac{c}{c'}$? Are the sides proportional?

290. Special terms in a proportion. In the proportion $\frac{a}{b} = \frac{c}{d}$ the last term, d, is called the **fourth proportional.**

In the proportion $\frac{a}{b} = \frac{b}{c}$ the last term, c, is called the **third proportional,** and b is called the **mean proportional.**

291. Axiom 12. *Like powers or like positive roots of equals are equal.*

Ex. 1. What is the mean proportional between 4 and 25? 1 and 36? 12 and 27? a^2 and b^2? πR^2 and $4\pi R^2$?

Ex. 2. Find the fourth proportional to 2, 5, and 6; 3, 8, and 9; $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{8}$; a, b, and c.

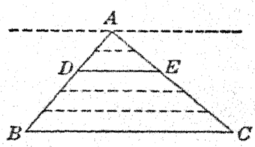
Ex. 3. Find the third proportional to 4 and 6; 4 and 10; 3 and 6.

Ex. 4. If $\frac{a}{b} = \frac{c}{d}$, is $\frac{a^2}{b^2} = \frac{c^2}{d^2}$? Why?

Ex. 5. If $\frac{a}{b} = \frac{b}{c}$, is $b = \sqrt{ac}$? Why?

Historical Note. — The use of proportion for the solution of problems was once called solution by the. **rule of three.** One of the parts of the Chinese mathematical work *K'iu-ch'ang Suan-shu* or *Arithmetic in Nine Sections* (about 1000 B.C.) was devoted to the solution of problems by this method.

292. Lines parallel to the base of a triangle. Draw any triangle and divide side AB into five equal parts. (See § 162.) Through the points of division draw lines parallel to BC. Why are the segments on AC equal? What is the ratio of AD to DB? Of AE to EC? Is $\frac{AD}{DB} = \frac{AE}{EC}$?

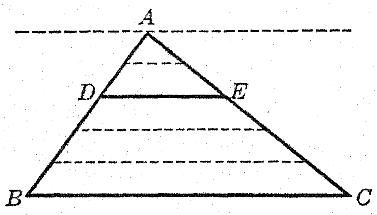


When two sides of a triangle are divided into segments so that the ratio of one pair of segments is equal to the ratio of the other pair, the sides are said to be divided proportionally.

If a line DE is parallel to BC in $\triangle ABC$, by finding a common unit of measure for AD and DB, and drawing parallels, can you prove the sides are divided proportionally? If the unit is contained in AD two times and in DB three times, what is the ratio of AD to DB? Will AE and EC be divided into equal segments? What will be the ratio of AE to EC? Then what proportion can be formed?

Proposition 1. Theorem

293. *A line parallel to one side of a triangle and intersecting the other two sides divides those two sides proportionally.*



Given: Triangle ABC, $DE \parallel BC$, AD and DB commensurable.

To prove: $AD : DB = AE : EC$.

Plan: With a unit which is contained a whole number of times in each, measure AD and DB. Then draw parallels to BC through the points of division. By § 152 show that the ratio of AE to EC is the same as the ratio of AD to DB.

Proof: Write the proof in full.

Note. — Two segments are said to be **commensurable** when there is a common unit contained in each a whole number of times.

294. Corollary. *One side is to either of its segments as the other side is to the corresponding segment.*

From the conclusion in § 293 we have, by addition:

$$\frac{AD + DB}{DB} = \frac{AE + EC}{EC} \quad \text{or} \quad \frac{AB}{DB} = \frac{AC}{EC}; \text{ and also}$$

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE} \quad \text{or} \quad \frac{AB}{AD} = \frac{AC}{AE}.$$

295. Some other proportions can also be developed from the conclusions in § 293 and § 294. Explain how the following are derived from § 293.

Ex. 1. $\frac{DB}{AD} = \frac{EC}{AE}$ **Ex. 2.** $\frac{AD}{AE} = \frac{DB}{EC}$

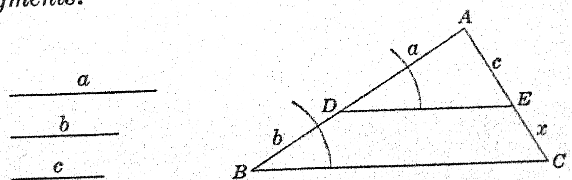
Explain how to obtain the following from § 294.

Ex. 3. $\frac{AB}{AC} = \frac{DB}{EC}$ **Ex. 4.** $\frac{AB}{AC} = \frac{AD}{AE}$

296. If a line is parallel to one side of a triangle we shall assume that "divided proportionally" means, as in § 293, $AD : DB = AE : EC$, or any of the forms derived in §§ 294–295.

Construction XVIII

297. *To construct the fourth proportional to three given line segments.*



Given: Segments a, b, and c.

Required: Construct the fourth proportional to a, b, and c. Construction and proof left for you to write out.

EXERCISES

In the figure of § 293:

1. If $AD = 10$ in., $DB = 8$ in., and $AE = 6$ in., find EC.
2. If $AD = 6$ in., $DB = 9$ in., and $EC = 5$ in., find AE.
3. If $AE = 5$ in., $DB = 12$ in., and $EC = 3$ in., find AD.
4. If AD is half of BD, and $AE = 6$ in., find EC.

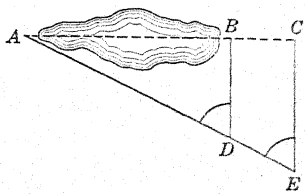
5. If $EC = 2\ AE$, and $BD = 8$ in., find AB.

6. If $AE = \frac{1}{3}\ EC$, and $AB = 16$ in., find AD and DB.

7. If $AB = 10$ in., $AC = 15$ in , and $AE = 6$ in., find DB.

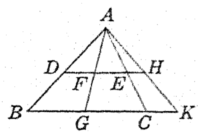
8. Construct the fourth proportional to three segments which are 1 in., 2 in., and $1\frac{1}{2}$ in. long. Measure the resulting segment and compare with the computed result.

9. Construct the third proportional to two segments which are $\frac{5}{8}$ in. and $1\frac{1}{4}$ in. in length. Compare the measured result with that found by computation.



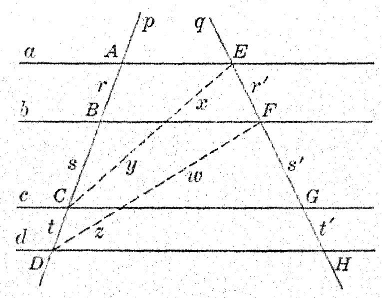
10. Some boy scouts measured the length of a pond AB by constructing a triangle ACE and taking BD parallel to CE as indicated by the drawing. Explain what measurements were necessary.

11. If $AE = 200$ yd., $DE = 60$ yd., $BC = 40$ yd., how long is AB?



12. In $\triangle ABC$ with $DE \parallel BC$, prove that any transversal from A to point G in BC will be divided by DE at F into segments so that $AD : DB = AF : FG$.

13. In the figure of Ex. 12, prove that if AK is cut by DE produced in H, and by BC produced in K, then $AF : FG = AH : HK$.



***14.** If lines a, b, c, and d are parallel, and AD and EH are any two transversals, prove that $r : s = r' : s'$.

HINT. — In $\triangle ACE$, $b \parallel AE$; and in $\triangle ECG$, $b \parallel CG$.

***15.** In the same figure prove that $s : t = s' : t'$.

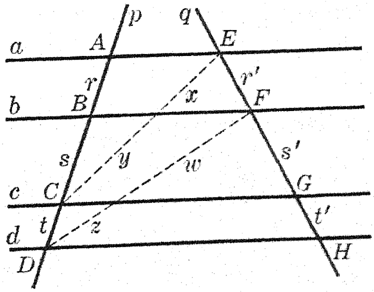
HINT. — Use $\triangle\!\!\triangle$ DFB and FDH.

***16.** From the results of Ex. 14 and 15 prove that

$$\frac{r}{r'} = \frac{s}{s'} = \frac{t}{t'}$$

PROPOSITION 2. THEOREM

298. *The segments cut off on two transversals by a series of parallels are proportional.*



Given: Parallels a, b, c, and d cut by transversals p and q into segments r, s, t, r', s', and t', respectively.

To prove: $\frac{r}{r'} = \frac{s}{s'} = \frac{t}{t'}$.

Plan: Draw transversals CE and DF forming triangles as shown. In triangle ACE, $\frac{r}{s} = \frac{x}{y}$, and in triangle CEG, $\frac{r'}{s'} = \frac{x}{y}$. Why? Hence, $\frac{r}{s} = \frac{r'}{s'}$, or $\frac{r}{r'} = \frac{s}{s'}$. Similarly, using triangles DHF and DFB show that $\frac{s}{s'} = \frac{t}{t'}$.

Proof: Write the proof in full.

EXERCISES

For the following exercises use the figure in § 298.

1. If $r = 4$ in., $s = 5$ in., $t = 2$ in., and $r' = 5$ in., find s' and t'.

2. If $r = 8$ in., $s = 10$ in., $t = 5$ in., and $s' = 15$ in., find r' and t'.

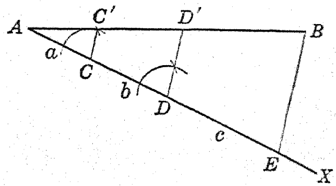
3. If $r = 4$ in., $s = 5$ in., $t = 2$ in., and $EH = 16\frac{1}{2}$ in., find r', s', and t'.

CONSTRUCTION XIX

299. *To divide a segment into parts proportional to any number of given segments.*

Given: Segment AB, and segments a, b, and c.

Required: Divide AB into segments proportional to a, b, and c.



Construction:

1. Draw AX, making any convenient angle with AB.

2. On AX take $AC = a$, $CD = b$ and $DE = c$.

3. Draw BE, and construct CC' and $DD' \parallel BE$.

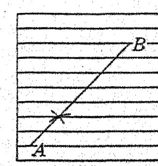
4. Segments AC', $C'D'$, $D'B$ are proportional to a, b, and c.

EXERCISES

1. Divide a given segment into parts whose ratio is 5 to 8.

2. Divide a segment into two parts whose ratio shall equal $2\frac{1}{4}$.

3. Divide a segment $2\frac{3}{4}$ in. long into parts whose ratio is $2:3$ as follows. Draw any convenient angle with vertex A. On one side of the angle take a segment $AB = 2\frac{3}{4}$ in. On the other side of the angle mark off any convenient unit five times so that $AM = MP = PQ = QR = RC$. Draw BC, and draw a line through $P \parallel BC$, cutting AB in P'. Prove that $AP' : P'B = 2:3$.



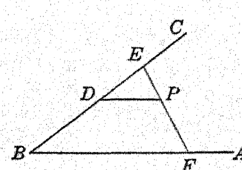
4. Can you explain from the figure how a piece of ruled paper can be used to divide a segment AB into parts whose ratio is $2:5$.

5. Using the method of Ex. 4, divide a segment $2\frac{7}{8}$ in. long into parts whose ratio is 1 to 5.

---

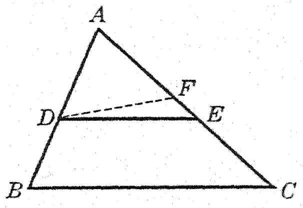
6. Having given a point P within $\angle ABC$, draw a line through P so that the segment of it lying within the angle shall be divided at P in the ratio of 1 to 2.

SUGGESTION. — If $EP : PF = 1:2$, what relation must exist between ED and DB if $PD \parallel AB$?



Proposition 3. Theorem

300. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given: Triangle ABC with DE drawn so that $AB : DB = AC : EC$.

To prove: $DE \parallel BC$.

Plan: Use the indirect method. If DE is not parallel to BC, and DF is, what proportion is there between the segments DF intercepts on AB and AC? Compare this proportion with the proportion given and use § 289–*5*.

Proof:

STATEMENTS	REASONS
1. If DE is not $\parallel BC$, suppose that DF is.	1. *Post.* 12.
2. Then $AB : DB = AC : FC$.	2. § 296.
3. But $AB : DB = AC : EC$.	3. *Given.*
4. $\therefore EC = FC$.	4. § 289–*5*.
5. Then since F coincides with E, DE coincides with DF.	5. *Post.* 3.
6. $\therefore DE \parallel BC$.	6. *DE coincides with DF.*

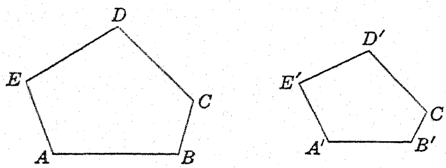
Ex. 1. If $AD = 12$ in., $DB = 9$ in., $AE = 8$ in., and $EC = 6$ in., is $DE \parallel BC$?

Ex. 2. What relation does the theorem in § 300 have to the theorem in § 293?

301. Corresponding sides and angles. Two polygons which have the angles of one equal, respectively, to the angles of the other, are called **mutually equiangular.**

The pairs of equal angles are called **corresponding angles.** The sides included between corresponding angles are called **corresponding sides.**

302. Similar ($\sim$) polygons. Similar polygons are those which (1) are mutually equiangular and (2) have their corresponding sides proportional.



SIMILAR POLYGONS

Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

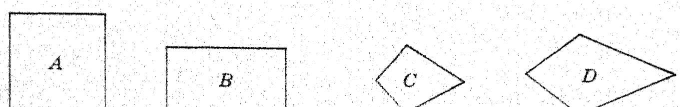
(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$; and

(2) $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$$

303. The ratio of two corresponding sides is called the **ratio of similitude.**

EXERCISES

1. Polygons A and B are mutually equiangular. Are their corresponding sides proportional? Are they similar?



Polygons C and D have their sides proportional. Are they mutually equiangular? Are they similar?

2. Are any two squares similar? Any two rectangles? Any two rhombuses?

3. Is a square ever similar to a rhombus?

4. Are any two equilateral triangles similar? Any two isosceles triangles? Any two right triangles?

5. Divide a line $3\frac{7}{8}$ in. long into segments proportional to 1, 3, and 4.

6. Construct the fourth proportional to segments a, b, and c.

7. Construct the third proportional to a and b.

8. Construct a segment x so that $x = bc \div a$.

9. Construct a segment x so that $x = b^2 \div a$.

10. Construct a segment x so that $cx = ab$.

11. Construct a segment x so that $x \div c = a \div b$.

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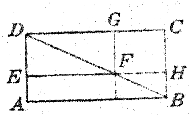
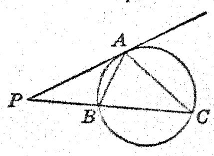
12. Prove that a line parallel to the bases of a trapezoid intercepts proportional segments on the non-parallel sides and on the diagonals.



13. $ABCD$ and $EFGD$ are rectangles, F being a point on diagonal DB. Prove that the two rectangles are similar.

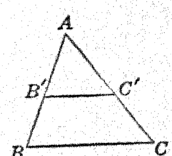
14. In Ex. 13 if $AE = \frac{1}{3}\,ED$, what is the ratio of similitude?



15. PA is a tangent and PBC a secant. Prove that $\triangle PAC$ and PAB are mutually equiangular.

16. Prove the theorem in § 300 by assuming that if DE is not $\parallel$ BC, it is parallel to BG which intersects AC produced in G.

304. Mutually equiangular triangles. If $\triangle ABC$ and $A'B'C'$ are mutually equiangular, what must we prove to show that they are similar (§ 302)?

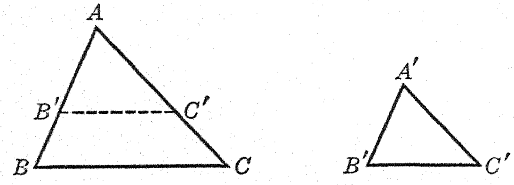


If the triangles are placed as shown in the figure, can you prove $B'C' \parallel BC$? What proportion follows?

Place the triangles again so that $\angle B'$ coincides with $\angle B$, and see if you can prove that two triangles are similar if they have two angles of one equal, respectively, to two angles of the other.

Proposition 4. Theorem

305. *Two triangles are similar if two angles of one are equal, respectively, to two angles of the other.*



Given: $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$ and $\angle B = \angle B'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

Plan: Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with its equal $\angle A$. Then show that $B'C' \parallel BC$, and use § 293. Repeat, placing $\angle B'$ on $\angle B$.

Proof:

STATEMENTS	REASONS
1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$. Then, since $\angle B = \angle B'$, $B'C' \parallel BC$.	1. *Given and* § 119.
2. $AB : AB' = AC : AC'$ or $AB : A'B' = AC : A'C'$.	2. § 293 *and Ax.* 7.
3. In the same way, by placing $\triangle A'B'C'$ so that $\angle B'$ coincides with $\angle B$ it can be proved that $AB : A'B' = BC : B'C'$.	3. *By reasoning as in* 1 *and* 2.
4. $\therefore AB : A'B' = AC : A'C' = BC : B'C'$.	4. *Ax.* 1.
5. $\angle C = \angle C'$.	5. § 126.
6. $\therefore \triangle ABC \sim \triangle A'B'C'$.	6. § 302.

306. Corollary. *Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.*

EXERCISES

In Ex. 1–3, similar triangles ABC and $A'B'C'$ have corresponding sides a, b, c, and a', b', c', respectively.

1. $a = 8$ in., $b = 11$ in., $c = 9$ in., $a' = 4$ in. Find b' and c'.

2. $a = 12$ in., $b = 15$ in., $a' = 3$ in., $c' = 5$ in. Find c and b'.

3. $a = 27$ ft., $c = 30$ ft., $b' = 6$ in., $c' = 10$ in. Find b and a'.

4. The sides of a triangle are 2 in., 3 in., and 4 in. The shortest side of a similar triangle is 5 in. Find the other two sides.

5. The base of an isosceles triangle is 18 in. and one of the equal sides is 24 in. Find the sides of a similar triangle whose greatest side is 8 in.

---

6. If an acute angle of one right triangle is equal to an acute angle of another, the triangles are similar.

7. If two triangles have their sides respectively parallel, or, respectively perpendicular, the triangles are similar.

8. A line parallel to the base of a triangle forms a triangle similar to the given triangle.

9. Two isosceles triangles are similar if a base angle of one equals a base angle of the other.

10. Two isosceles triangles are similar if their vertex angles are equal.

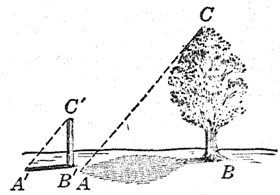
11. Are two isosceles triangles always similar if an angle of one equals an angle of the other?

12. If each of two triangles is similar to a third triangle, they are similar to each other.

13. Corresponding angle bisectors of similar triangles have the same ratio as any two corresponding sides.

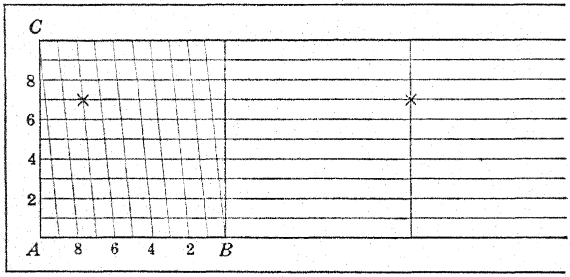
PRACTICAL APPLICATIONS

(OPTIONAL)



1. A stick $B'C'$ held vertically casts a shadow $A'B'$ which is $3\frac{1}{2}$ ft. long. At the same time a tree BC casts a shadow AB, which is 50 ft. long. If $B'C' = 6$ ft., find BC.

2. The instrument shown in this drawing is a *diagonal scale*, used by draftsmen for measuring very short lengths. $AB = 1$ in. AB and AC are each divided into 10 equal parts, and parallels are drawn through the



points of division as shown. By means of this scale distances can be measured to hundredths of an inch.

Find .01 in. on this scale; .02 in.; .03 in.; .04 in., etc.

Upon what theorem does this depend?

3. Find .12 in. on the diagonal scale; .35 in.; .93 in.; .68 in.

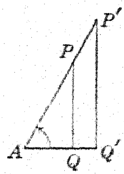
4. What is the distance between the points marked $x \cdots x$ on the diagonal scale?

5. Find 1.63 in. on the diagonal scale; 1.85 in.; 1.19 in.

6. The fact that the heights of two objects are to each other as the lengths of their shadows has been used since the time of the ancient Greeks. Thales (about 600 B.C.) is said to have amazed the Egyptians by measuring the heights of the pyramids by the lengths of their shadows.

Give a proof of this principle.

307. A. Trigonometric ratios. Some very interesting ratios follow from similar triangles. If from any two points P and P' on a side of $\angle A$ perpendiculars are dropped to the other side, the triangles formed are similar. Why? We have, therefore:



$$\frac{QP}{AP} = \frac{Q'P'}{AP'}; \quad \frac{AQ}{AP} = \frac{AQ'}{AP'}; \quad \text{and} \quad \frac{QP}{AQ} = \frac{Q'P'}{AQ'}.$$

Since these ratios are the same for any angle A, no matter where points P and P' may be, we have named them the sine, cosine, and tangent of angle A.

If we call the side PQ the *side opposite* angle A, the side AQ the *side adjacent* to angle A, and AP the *hypotenuse*, we have:

$$\text{sine } A = \frac{\text{side opp.}}{\text{hyp.}}, \quad \text{cosine } A = \frac{\text{side adj.}}{\text{hyp.}}, \text{tangent } A = \frac{\text{side opp.}}{\text{side adj.}}.$$

The abbreviation of sine is *sin*; of cosine is *cos*; and of tangent is *tan*.

308. A. How to find the value of the trigonometric ratios. To find the value of any of these ratios approximately you can measure the sides PQ, AQ, and AP and divide. By making your figure fairly large you can find the value to at least one decimal place. However, the table on page 298 gives the values to three decimals.

To find the value of an angle when the ratio is given, look for the value of the ratio under the proper heading at the top of the page and read the angle in the column at the left.

Historical Note. — While Thales, in measuring the height of a pyramid, used the ratio between its height and its shadow, which corresponds to the tangent, he did not definitely connect this ratio with the angle.

An Arab, Abû'l-Wefa, about the year 1000, made the first table of tangents. Tangents were called in Latin *umbra versa*, meaning *the turned shadow*.

EXERCISES

Check these values from the table:

1. $\sin 3° = 0.052$
2. $\tan 7° = 0.123$
3. $\cos 20° = 0.940$
4. $\tan 54° = 1.376$
5. $\sin 73° = 0.956$
6. $\cos 84° = 0.105$

Find the value of:

7. $\sin 38°$
8. $\tan 13°$
9. $\cos 62°$
10. $\cos 18°$
11. $\sin 41°$
12. $\tan 65°$
13. $\tan 43°$
14. $\sin 69°$
15. $\cos 45°$

Find the value of angle x if:

16. $\sin x = 0.857$
17. $\tan x = 0.510$
18. $\tan x = 2.475$
19. $\cos x = 0.500$
20. $\sin x = 0.500$
21. $\cos x = 0.866$
22. $\sin x = 0.995$
23. $\tan x = 1.000$
24. $\cos x = 0.857$

309. A. Interpolation. We interpolate when we find values of functions between those given in the table.

To find the value of a ratio not given in the tables. Find $\tan 42° 35'$. Proceed as follows:

$$\tan 42° = 0.900$$
$$\tan 43° = 0.933$$

Then since $42° 35'$ lies $\frac{35}{60}$th of the way between 42° and 43°, we have

$$\tan 42° 35' = \tan 42° + \tfrac{35}{60} \times (0.933 - 0.900), \text{ or}$$
$$\tan 42° 35' = 0.900 + \tfrac{35}{60} \times 0.033 = 0.900 + 0.019.$$

Thus, $\tan 42° 35' \doteq 0.919$.

To find the value of an angle when its ratio is not given in the table. Find $\angle A$ when $\sin A = 0.667$.

$\sin 41° = 0.656$
$\sin 42° = 0.669$

Since 0.667 lies between 0.656 and 0.669, $\angle A$ must lie between 41° and 42°. Thus

$$\angle A = 41° + \frac{0.667 - 0.656}{0.669 - 0.656} \times 60' = 41° + \frac{0.011}{0.013} \times 60'$$
$$= 41° + 51' = 41° \, 51'$$

EXERCISES

Check these values:

1. $\sin 48° \, 15' = 0.746$

2. $\tan 32° \, 45' = 0.643$

3. $\cos 27° \, 30' = 0.887$

4. $\tan 57° \, 10' = 1.550$

5. $\sin 9° \, 40' = 0.168$

6. $\cos 29° \, 20' = 0.872$

Find:

7. $\tan 18° \, 30'$

8. $\tan 61° \, 45'$

9. $\sin 5° \, 15'$

10. $\sin 43° \, 50'$

11. $\cos 26° \, 30'$

12. $\cos 29° \, 55'$

13. $\tan 50° \, 5'$

14. $\cos 41° \, 10'$

15. $\sin 75° \, 40'$

Find angle x if:

16. $\sin x = 0.600$

17. $\cos x = 0.800$

18. $\tan x = 0.443$

19. $\cos x = 0.862$

20. $\sin x = 0.531$

21. $\tan x = 0.667$

22. $\tan x = 1.611$

23. $\tan x = 1.840$

24. $\cos x = 0.495$

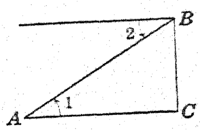
25. $\cos x = 0.947$

Historical Note. — Although there are some earlier traces of the use of trigonometry, it is probably true that the real beginning of this study was made by Hipparchus, the greatest of Greek astronomers, who, it is said, wrote twelve books on the computation of *chords* of angles (140 B.C.). These probably were the first trigonometric tables.

VALUES OF SINES, COSINES, AND TANGENTS

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	0.017	1.000	0.017	45	0.707	0.707	1.000
2	0.035	0.999	0.035	46	0.719	0.695	1.036
3	0.052	0.999	0.052	47	0.731	0.682	1.072
4	0.070	0.998	0.070	48	0.743	0.669	1.111
5	0.087	0.996	0.087	49	0.755	0.656	1.150
6	0.105	0.995	0.105	50	0.766	0.643	1.192
7	0.122	0.993	0.123	51	0.777	0.629	1.235
8	0.139	0.990	0.141	52	0.788	0.616	1.280
9	0.156	0.988	0.158	53	0.799	0.602	1.327
10	0.174	0.985	0.176	54	0.809	0.588	1.376
11	0.191	0.982	0.194	55	0.819	0.574	1.428
12	0.208	0.978	0.213	56	0.829	0.559	1.483
13	0.225	0.974	0.231	57	0.839	0.545	1.540
14	0.242	0.970	0.249	58	0.848	0.530	1.600
15	0.259	0.966	0.268	59	0.857	0.515	1.664
16	0.276	0.961	0.287	60	0.866	0.500	1.732
17	0.292	0.956	0.306	61	0.875	0.485	1.804
18	0.309	0.951	0.325	62	0.883	0.469	1.881
19	0.326	0.946	0.344	63	0.891	0.454	1.963
20	0.342	0.940	0.364	64	0.899	0.438	2.050
21	0.358	0.934	0.384	65	0.906	0.423	2.145
22	0.375	0.927	0.404	66	0.914	0.407	2.246
23	0.391	0.921	0.424	67	0.921	0.391	2.356
24	0.407	0.914	0.445	68	0.927	0.375	2.475
25	0.423	0.906	0.466	69	0.934	0.358	2.605
26	0.438	0.899	0.488	70	0.940	0.342	2.747
27	0.454	0.891	0.510	71	0.946	0.326	2.904
28	0.469	0.883	0.532	72	0.951	0.309	3.078
29	0.485	0.875	0.554	73	0.956	0.292	3.271
30	0.500	0.866	0.577	74	0.961	0.276	3.487
31	0.515	0.857	0.601	75	0.966	0.259	3.732
32	0.530	0.848	0.625	76	0.970	0.242	4.011
33	0.545	0.839	0.649	77	0.974	0.225	4.331
34	0.559	0.829	0.675	78	0.978	0.208	4.705
35	0.574	0.819	0.700	79	0.982	0.191	5.145
36	0.588	0.809	0.727	80	0.985	0.174	5.671
37	0.602	0.799	0.754	81	0.988	0.156	6.314
38	0.616	0.788	0.781	82	0.990	0.139	7.115
39	0.629	0.777	0.810	83	0.993	0.122	8.144
40	0.643	0.766	0.839	84	0.995	0.105	9.514
41	0.656	0.755	0.869	85	0.996	0.087	11.430
42	0.669	0.743	0.900	86	0.998	0.070	14.301
43	0.682	0.731	0.933	87	0.999	0.052	19.081
44	0.695	0.719	0.966	88	0.999	0.035	28.636
45	0.707	0.707	1.000	89	1.000	0.017	57.290

310. A. Angles of elevation and depression. Angle 1, the angle between the horizontal line AC and the line of sight AB, is called the **angle of elevation;** $\angle 2$ is called the **angle of depression.**



EXERCISES

1. What trigonometric function would you use to find the *side opposite* in a right triangle, if the acute angle and the *side adjacent* are given?

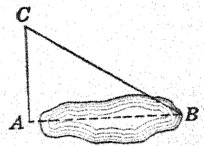
2. What function would you use to find the *side opposite* when the acute angle and *hypotenuse* are given?

3. How would you find the acute angle if the *side opposite* and the *side adjacent* are given? The *side opposite* and the *hypotenuse*? The *side adjacent* and the *hypotenuse*?

4. If $\dfrac{x}{50} = \tan 43°$, find x.

5. If $\dfrac{150}{x} = \sin 38°$, find x.

6. The angle of elevation of a church steeple from a point 100 ft. away is 36°. How high is the steeple?



7. In order to find the distance from A to B across a lake, a surveyor measured a line AC at right angles to AB, then measured $\angle ACB$. If $AC = 820$ ft. and $\angle ACB = 56°$, find AB.

8. The angle of elevation of the top of a tree from a point 80 ft. from the foot of the tree is 41°. Find the height of the tree.

9. The sides of a right triangle are 6 in., 8 in., and 10 in. Find the angle included between the 8 in. and 10 in. sides.

10. The distance from the observer to the foot of a monument is 275 ft. The angle of elevation of the top of the monument at the point of observation is 52°. Find the height.

11. Find the base angles of an isosceles triangle when one of the equal sides is 15 in. and the altitude is 10 in.

12. When a man 5 ft. 11 in. tall casts a shadow 8 ft. 7 in. long, what is the angle of elevation of the sun?

13. What is the angle of elevation of the sun when a man's shadow is twice his own height?

14. The top of a building known to be 136 ft. high forms an angle of elevation of 23° at a point of observation. How far is the observer from the building?

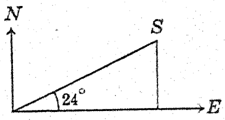
15. The Empire State Building in New York City is 1024 ft. high. What is its angle of elevation from a point in New York Bay, $2\frac{1}{2}$ miles away?

16. The side opposite an acute angle A of a right triangle is 12 in. and the adjacent side is 20 in. Find $\angle A$.

17. A vertical rod 8 ft. high casts a shadow 3 ft. 6 in. long. Find the angle of elevation of the sun.

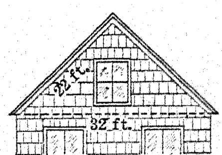
18. The distance from the base to the top of a hill, up a uniform incline of 40°, is 300 yd. What is the altitude of the top above the base?

19. A ship is sailing from New York in a direction 24° north of east. When the log shows that it has gone 450 mi., how far is the ship east of New York? How far north?



20. The vertex angle of an isosceles triangle is 47°. One equal side is 18 in. Find the altitude on the base.

21. A house 32 ft. wide has a gable with rafters 22 ft. long, excluding the part which projects below the eaves. Find the *pitch* of the roof, or the angle between a rafter and the horizontal.

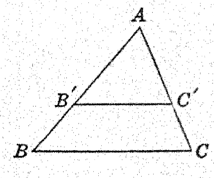


22. A boat has sailed 20 mi. in a northeasterly direction. How far east has it gone? How far north?

311. A challenge. Perhaps you can prove the next theorem without the text.

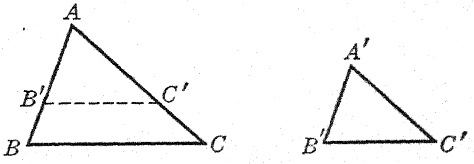
If $\triangle ABC$ and $A'B'C'$ have $\angle A = \angle A'$ and $AB : A'B' = AC : A'C'$, can you prove that they are similar?

Place the triangles as shown in the figure. Is $B'C' \parallel BC$? Can you prove the triangles mutually equiangular?



Proposition 5. Theorem

312. *Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.*



Given: Triangles ABC and $A'B'C'$, with $\angle A = \angle A'$, and $AB : A'B' = AC : A'C'$.

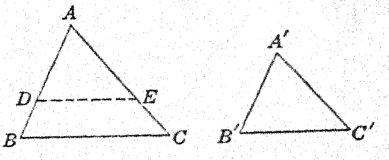
To prove: $\triangle ABC \sim \triangle A'B'C'$.

Plan: Will superposing $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$ enable you to prove $B'C' \parallel BC$? (§ 300.)

Proof: Write out the proof.

313. Triangles with their sides proportional. See if you can prove the next theorem after studying these exercises.

1. In $\triangle$ ABC and $A'B'C'$ given $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$, $AD = A'B'$, and $AE = A'C'$. Prove $DE \parallel BC$.



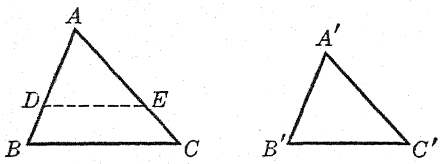
2. In Ex. 1 prove $\triangle ADE \sim \triangle ABC$.

3. Using the results of Ex. 1 and 2, can you prove $\triangle ADE \cong \triangle A'B'C'$?

Hint. — It will be sufficient to prove $DE = B'C'$. Why? Why is $AB : AD = BC : DE$? Why is $AB : A'B' = BC : B'C'$? Then why is $DE = B'C'$?

Proposition 6. Theorem

314. *Two triangles are similar if their corresponding sides are proportional.*



Given: Triangles ABC and $A'B'C'$ with $AB : A'B' = AC : A'C' = BC : B'C'$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

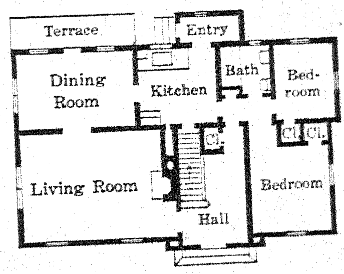
Plan: Why cannot you superpose $\triangle A'B'C'$ on $\triangle ABC$? Construct $\triangle ADE$, making $AD = A'B'$ and $AE = A'C'$, and prove $\triangle ADE \sim \triangle ABC$. Then prove $\triangle ADE \cong \triangle A'B'C'$.

Proof:

STATEMENTS	REASONS
1. On AB take $AD = A'B'$ and on AC take $AE = A'C'$. Draw DE. Since $AB : A'B' = AC : A'C'$, $AB : AD = AC : AE$.	1. *Given and Ax.* 7.
2. $\triangle ADE \sim \triangle ABC$.	2. § 312.
3. $\therefore AB : AD = BC : DE$.	3. § 302.
4. $AB : A'B' = BC : B'C'$ or $AB : AD = BC : B'C'$.	4. *Given and Ax.* 7.
5. $\therefore DE = B'C'$.	5. § 289–5.
6. $\therefore \triangle ADE \cong \triangle A'B'C'$.	6. § 80.
7. $\therefore \triangle ABC \sim \triangle A'B'C'$.	7. *Ax.* 7.

Ex. 1. In the figure of § 314: If $AB = 12$ in., $AC = 8$ in., and $BC = 10$ in., and $A'B' = 8$ in., find $B'C'$ and $A'C'$. If $BD = \frac{2}{5}\ AD$, what is the ratio of similitude?

315. Maps and plans. A **map** or **plan** is a figure similar to the figure formed by the object which it represents. Thus, a map of a state is a drawing similar to the figure formed by the state itself. The drawing shows an architect's *floor plan* of a house.



A map or plan is always *drawn to scale*, i.e. in the map or plan the distances are made *proportional* to the actual distances which they represent.

Thus, a map of the United States which is drawn so that 200 miles measured anywhere across the country is represented on the map by a distance of 1 inch, is said to be *drawn to the scale of 200 miles to an inch* (scale: 200 mi. = 1 in.).

EXERCISES

1. Consult a map of the United States. Find the scale to which it is drawn. Find from this map the number of miles in a straight line from Boston to San Francisco.

2. A map of Illinois drawn to the scale of 200 mi. to an inch is $1\frac{5}{8}$ in. long. How many miles long is the state?

3. On a map drawn to the scale of 240 mi. to an inch, the distance from Chicago to Denver is $3\frac{9}{16}$ in. How many miles is it from Chicago to Denver?

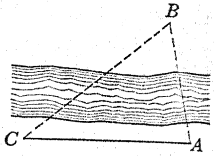
4. In the house of which the floor plan is shown in § 315, the scale is 2 ft. = $\frac{1}{16}$ in. What is the width of the living room?

5. In Ex. 4, determine the number of feet in the width of the dining room. Find the length.

6. How many feet wide is the hall of this house?

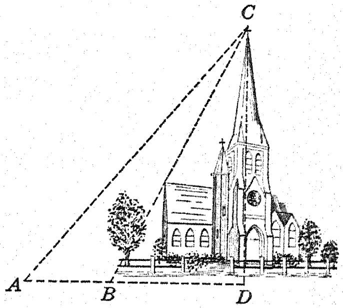
7. Draw a rectangle representing a rectangular field that is 1200 ft. long and 480 ft. wide to a scale of 240 ft. to an inch. What are the dimensions of the drawing?

8. The distance from A to the inaccessible point B may be obtained as follows: Measure a base line AC. Measure $\angle ACB$ and $\angle BAC$. Then construct a map of the measurements to scale, and determine from the map the distance from A to B.

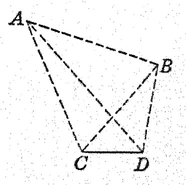


If $AC = 960$ ft., $\angle ACB = 40°$, and $\angle BAC = 75°$, draw a map of the measurements to the scale of 160 ft. to an inch, and compute AB from the map.

9. In order to find the height of a church spire CD, the base line AB is measured 75 ft. long toward the foot of the spire D. It is found that $\angle DAC = 50°$ and $\angle DBC = 80°$. Make a drawing of these measurements to the scale of 25 ft. to an inch, and compute the height CD of the spire.

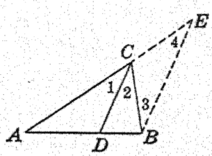


10. A and B are two forts in the lines of the enemy, and it is desired to know their distance apart and their distances from our lines. From point C in our lines, we measure $\angle DCA$ and $\angle DCB$. Then we go to a second point D and measure $\angle ADC$ and $\angle BDC$. $\angle DCA = 120°$, $\angle DCB = 50°$, $\angle ADC = 45°$, $\angle BDC = 100°$, and $CD = 2000$ ft. Draw a plan to the scale of 500 ft. to the inch, and find the distances AC, BD, and AB.



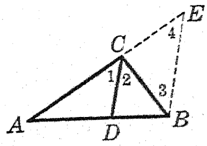
316. The bisector of an angle of a triangle. If CD bisects $\angle C$ in $\triangle ABC$, see if you can prove that $AD : DB = AC : CB$.

If BE is drawn parallel to CD, what proportion is there between the segments made by CD on AB and AE? Find the relations between the angles and show that $CB = CE$ (§ 76).



PROPOSITION 7. THEOREM

317. *The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Given: Triangle ABC, with CD bisecting $\angle C$.

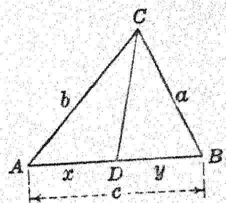
To prove: $AD : DB = AC : CB$.

Plan: If BE is drawn $\parallel CD$, what relation is there between segments of the sides of $\triangle ABE$ (§ 293)? Compare that proportion with what you are to prove, then find the relation between $\angle$s 1 and 4, 2 and 3, and 3 and 4.

Proof: For you to write.

EXERCISES

In Ex. 1–4, triangle ABC is given with CD the bisector of $\angle C$.



1. $a = 12$ in., $b = 16$ in., $c = 14$ in. Find x and y.

2. $a = 8$ in., $b = 10$ in., $c = 12$ in. Find x and y.

3. $a = 28$ in., $b = 32$ in., $x = 16$ in. Find c.

4. $a = 15$ in., $x = 12$ in., $y = 10$ in. Find b.

5. Is Prop. 7 true when $\triangle ACB$ is isosceles? What can you then say about CD?

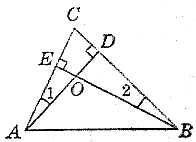
---

6. The diagonals of a trapezoid divide each other into proportional segments.

7. The corresponding medians of two similar triangles have the same ratio as any two corresponding sides.

8. The line joining the mid-points of two sides of a triangle forms a triangle which is similar to the given triangle.

9. AD and BE are altitudes of $\triangle ABC$. Prove $\triangle ADC \sim \triangle BCE$ and write the proportions that follow.



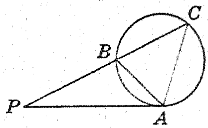
SUGGESTION. — Prove $\triangle$ ADC and BCE mutually equiangular. Then to get corresponding sides make a plan like the following.

(Corresponding sides are opposite equal angles.)

In $\triangle ADC$, side CD is opposite $\angle 1$; in $\triangle BCE$, side CE is opposite $\angle 2$. Since $\angle 1 = \angle 2$, CD and CE are corresponding sides.

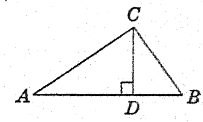
In $\triangle ADC$, side AC is opposite $\angle D$; in $\triangle BCE$, BC is opposite $\angle E$. Since $\angle D = \angle E$, AC and BC are corresponding sides.

In $\triangle ADC$, side AD is opposite $\angle C$; in $\triangle BCE$, BE is opposite $\angle C$. Since $\angle C = \angle C$, AD and BE are corresponding sides.



10. If PA is a tangent, prove that $\triangle PAC \sim \triangle PAB$, and write the resulting proportions.

11. If $\angle C$ is a right angle and $CD \perp AB$, prove $\triangle$ ABC, ACD, and BCD similar and write all the proportions.



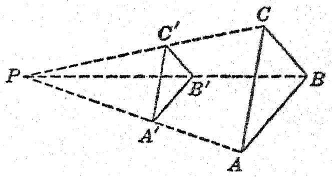
12. Chords AB and CD intersect at E within the circle. Prove $AE : CE = DE : BE$.

13. In triangle ABC altitudes BD and CE intersect at O. Prove $CO : OB = DO : OE$.

14. Rectangle $ABCD$ has side AD extended to E and BE is drawn intersecting diagonal AC in F. Prove $AE : BC = EF : FB$.

15. $\triangle ABC$ is inscribed in a circle. A tangent at the extremity D of diameter AD meets AB produced at E and AC produced at F. Prove $\triangle ABC \sim \triangle AEF$.

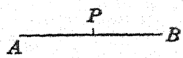
16. If AD and BE are altitudes of $\triangle ABC$, prove that $AD : BE = AC : BC$. That is, the altitudes are inversely proportional to their bases.

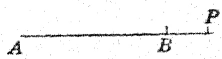


17. P is any point outside $\triangle ABC$. From any point A' on PA, $A'B'$ is drawn $\parallel AB$, $B'C' \parallel BC$, and $C'A'$ is drawn. Prove $C'A' \parallel CA$ and $\triangle A'B'C' \sim \triangle ABC$.

18. In Ex. 17, if $PA' = A'A$, what is the ratio of similitude of $\triangle$ ABC and $A'B'C'$?

318. Parts of a segment. You know that the point P divides the segment AB into two parts, AP and PB. Probably you have never thought of the point P as dividing a segment without being between A and B.





It is often convenient to say that P in the second figure divides the segment AB **externally** into the segments AP and PB, while in the first figure we say that P divides AB **internally.**

Notice that just as in the first figure, the segments are AP and PB measured from P to each end of the segment AB.

EXERCISES

1. If a segment AC is divided *internally* at B, what are the parts? What are the parts if it is divided *externally* at B? Draw a figure for each case.

2. AB is 6 in. long and is divided internally at P so that $AP : PB = \frac{1}{2}$. How long is PB? How long if $AP : PB = \frac{1}{5}$? $\frac{4}{5}$?

3. AB is 15 in. long and is divided externally at Q. How long is AQ if $AQ : QB$ is $\frac{2}{1}$? If it is $\frac{3}{2}$? If it is $\frac{8}{7}$?

---

4. A segment AB, 5 in. long, is divided internally at P so that AP is 3 in. long. If AB is divided externally at Q so that $AP : PB = AQ : QB$, find the length of AQ.

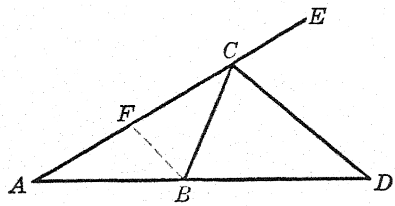
***5.** If, from any point P on a circle a perpendicular PD is drawn to the diameter AB, prove that $\triangle PAD \sim \triangle PDB \sim \triangle PAB$.

***6.** From the result in Ex. 5, prove that $\overline{PD}^2 = AD \cdot DB$.

***7.** Using the result found in Ex. 5, prove that $\overline{AP}^2 = AD \cdot AB$.

PROPOSITION 8. THEOREM

319. B. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*



Given: Triangle ABC with CD bisecting exterior $\angle BCE$ and dividing side AB externally into segments AD and DB.

To prove: $AD : DB = AC : CB$.

Plan: Think: "If I draw $BF \parallel CD$, I shall have the sides of $\triangle ACD$ divided proportionally" (§293). Then $CF = CB$. Why?

Proof: Write the proof.

320. A. **Harmonic division of a segment.** A segment is said to be **divided harmonically** when it is divided internally at P and externally at Q so that $AP : PB = AQ : QB$.

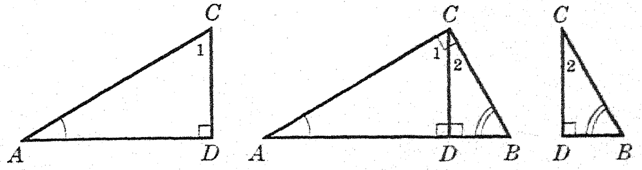
321. A. COROLLARY. *The bisector of the interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side of the triangle harmonically.*

Ex. 1. In $\triangle ABC$, $AB = 7$ in., $CA = 5$ in., and $BC = 6$ in. Find the segments made on each side by the bisector of the opposite angle.

Ex. 2. The sides of a triangle are 14 in., 18 in., and 20 in. Find the segments made on each side by the bisector of the opposite exterior angle.

PROPOSITION 9. THEOREM

322. *In any right triangle, the perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two triangles similar to the given triangle.*



Given: Triangle ABC, with $\angle C$ a right angle and $CD \perp AB$.

To prove: $\triangle ACD$ and $BCD \sim \triangle ABC$.

Plan: Prove the $\triangle$ mutually equiangular.

Proof: Left for you to prove.

323. COROLLARY 1. Under the conditions in § 322,

I. *The two triangles are similar to each other.*

II. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

III. *Either side is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to it.*

SUGGESTION. — Use the plan explained in Ex. 9, § 317.

324. COROLLARY 2. *The perpendicular to the diameter of a circle from any point on the circle (a) is the mean proportional between the segments of the diameter; and (b) the chord from that point to either extremity of the diameter is the mean proportional between the diameter and the segment of the diameter adjacent to that chord.*

HINT. — Recall that an angle inscribed in a semicircle is a right angle and use § 323.

EXERCISES

For a review of algebraic processes see the *Appendix*.

1. The bisector of $\angle A$ of $\triangle ABC$ divides side BC into segments of 3 in. and 4 in. If one of the other sides is 6 in., find the third side.

2. Prove by § 317 that the bisector of the vertex angle of an isosceles triangle bisects the base.

In the figure of § 322:

3. If $AB = 25$ in. and $AD = 16$ in., find AC and BC.

4. If $AB = 18$ in. and $AC = 15$ in., find AD.

5. If $AD = 12$ in. and $DB = 27$ in., find CD.

6. If $AB = 26$ ft. and $CD = 12$ ft., find AD and DB.

7. If $AD = 4$ ft. and $AB = 20$ ft., find CD, AC, and BC.

8. If $AC = 4$ in. and $AB = 8$ in., find BC, AD, and CD.

9. If $BC = 6$ in. and $AB = 12$ in., find AC, AD, and CD.

10. If $\angle A = 30°$, prove $AD : DB = 3 : 1$. (See § 160.)

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11. Prove that $\overline{AC}^2 : \overline{BC}^2 = AD : DB$.

12. If $BC = a$ and $AC = 3\,a$, prove that $AD = 9\,DB$.

13. If $CD = 12$ ft. and $AB = 25$ ft., find AD and BC.

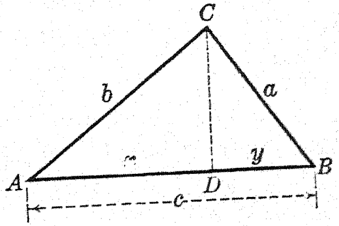
325. Pythagoras was a famous Greek philosopher and mathematician who lived about 540 B.C. He is believed to have given the first rigorous proof of the theorem that bears his name: *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*

The truth of the theorem for special cases was known by the Egyptians centuries before the time of Pythagoras, and was used by them in building their pyramids. They laid out perpendicular lines by stretching a rope around three pegs so placed that the distances between them were proportional to 3, 4, and 5. The same principle is employed today.

The proof of the **Pythagorean** (Pythag′o-rē′an) **theorem** given in the next section is based on § 323–III. This proof is attributed to the Hindus.

Proposition 10. Theorem

326. *In any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.*



Given: Triangle ABC, with sides a, b, and c, $\angle C$ a right angle.

To prove: $c^2 = a^2 + b^2$.

Plan: Use § 323–III.

Proof:

STATEMENTS	REASONS
1. Draw $CD \perp AB$ forming segments x and y on AB. $x : b = b : c, y : a = a : c$.	1. § 323–III.
2. $b^2 = cx, a^2 = cy$.	2. § 289–*1*.
3. $b^2 + a^2 = cx + cy = c(x + y)$.	3. *Ax.* 2.
4. $b^2 + a^2 = c^2$.	4. *Ax.* 7.

Note. — Another proof of the Pythagorean theorem is given in § 360. It was given in the first great geometry, *Euclid's Elements*, about 300 B.C. Still other proofs are suggested in the exercises following § 360. Just what proof was given by Pythagoras is not known.

EXERCISES

In the following set of exercises, c is the hypotenuse of a right triangle, and a and b are the legs.

1. If $a = 12$ in. and $b = 9$ in., find c.

2. If $a = 21$ ft. and $b = 20$ ft., find c.

3. If $c = 17$ ft. and $a = 8$ ft., find b.

4. If $a = 5$ in. and $c = 13$ in., find b.

5. If $a = 24$ in. and $b = 7$ in., find c.

6. If $a = 9$ in. and $c = 41$ in., find b.

7. The hypotenuse of a right triangle is 18 ft. and one leg is 14 ft. Compute the length of the other leg correct to the nearest tenth of a foot.

8. A baseball diamond is a square whose side is 90 ft. Find the length of the throw from first to third base.

9. Find correct to the nearest hundredth of an inch the diagonal of a square whose side is 4 in.

10. Find a formula for the diagonal of a square whose side is a.

11. Find the altitude of an equilateral triangle if each side is 12 in.

12. If each of the equal sides of an isosceles trapezoid is 61 ft., one base 40 ft., and the other base 62 ft., find the altitude.

13. How long a rope is required to reach from the top of a tent pole 12 ft. high to a peg in the ground 15 ft. from the foot of the pole?

14. The gable of a house is to be made 36 ft. wide and 12 ft. high above the eaves. How long must the rafters be cut if they are to extend 18 in. below the eaves?

15. A piece of cloth 27 in. wide is to be cut on the bias, at an angle of 45° with the edge. How long will the bias cut be?

16. An ancient Chinese problem: "A pool of water was 10 ft. across, and in the middle of it stood a reed which projected one foot above the water. When the wind blew the reed over, the top just reached to the edge of the pool. How deep was the water?"

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17. The diameters of two concentric circles are 84 in. and 85 in., respectively. Find the length of a chord of the larger that is tangent to the smaller.

18. The radius of a circle is 28 in. Find to tenths of an inch the length of the shortest chord that can be drawn through a point 6 in. from the center. (The shortest chord through a point is perpendicular to the diameter through that point.)

19. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides equals the sum of the squares of the other pair.

20. In any rectangle the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.

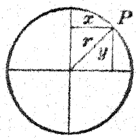
21. In rt. $\triangle ABC$, $AB = \frac{1}{4}m^2 + 1$, $AC = \frac{1}{4}m^2 - 1$, and $BC = m$. Show that $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

22. Show that $a^2 - b^2$, $a^2 + b^2$, and $2\,ab$ can represent numerically the sides of a right triangle.

23. Pythagoras showed that n, $\frac{n^2-1}{2}$, and $\frac{n^2+1}{2}$ can represent the numerical values of the sides of a right triangle. Verify this.

24. Proclus, a Greek mathematician who lived about 460 B.C., showed that the sides of a right triangle can be expressed by the literal numbers $2\,n + 1$, $2\,n^2 + 2\,n$, and $2\,n^2 + 2\,n + 1$. Verify.

25. For the same purpose Plato used the formulas $2\,n$, $n^2 - 1$, and $n^2 + 1$. Show that they are Pythagorean numbers.

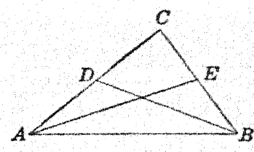


26. Show that if two perpendicular lines are drawn through the center of a circle, and x and y are the perpendicular distances of any point P of the circle from these lines, and r is the radius, then $x^2 + y^2 = r^2$.

***27.** The lengths of the radii of two circles are 10 in. and 6 in., respectively, and the distance between their centers 20 in. Find the lengths of their common external tangents. (See § 223.)

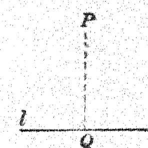
***28.** If medians are drawn from the extremities of the hypotenuse of a right triangle, four times the sum of the squares of the medians is equal to five times the square of the hypotenuse.

To prove that $4\,(\overline{AE}^2 + \overline{BD}^2) = 5\,\overline{AB}^2$.

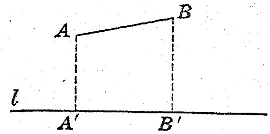


PROJECTION

327. Meaning of projection. By the **projection of a point** P on a line l we mean the foot of the perpendicular from P to l; that is, the point Q. The line PQ is used only to obtain the projection, Q.



The **projection of a segment** AB on a line l is $A'B'$, found by dropping perpendiculars, AA' and BB', from A and B, respectively, to l.



What relation do you think exists between the length of AB and $A'B'$? Are they ever equal? If AB is perpendicular to l, what is its projection on l?

NOTE. — A convenient abbreviation for "projection of AB on l" is p_l^{AB}.

EXERCISES

1. Draw an acute triangle with sides a, b, and c. Show by a drawing the projection of each side on each of the others.

2. Repeat Ex. 1, using an obtuse triangle.

3. If AB produced makes an angle of 60° with l, and $AB = 10$, find p_l^{AB}.

4. Can p_l^{AB} ever be greater than AB?

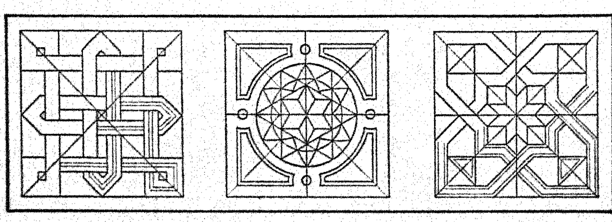
5. If AB produced makes an angle of 45° with l, and $AB = 16$, find p_l^{AB}.

6. If the equal sides of an isosceles trapezoid are each 10 and intersect a base at an angle of 60°, find the projection of one of the sides on the base.

7. In Ex. 6 find the projection if the angle is 30°; 45°.

8. Prove that the projection of the equal sides of an isosceles trapezoid on the base are equal.

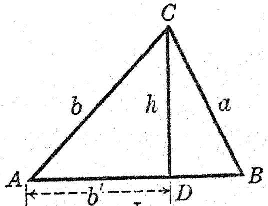
9. In $\triangle ABC$: Given $a = 6$, $p_c^a = 3$, find h_c; find $\angle B$.



SQUARE PANELS USED IN DECORATION

PROPOSITION 11. THEOREM

328. A. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides by the projection of the other side on it.*



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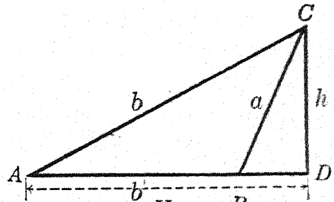


II

Given: Triangle ABC, with acute $\angle A$, side a opposite $\angle A$, b and c the other sides, b' the projection of b on c, h the altitude on c.

To prove: $a^2 = b^2 + c^2 - 2\,b'c$.

Plan: Express h in terms of the sides in both right $\triangle$. Then eliminate h.

Proof:

STATEMENTS	REASONS
1. $h^2 = b^2 - b'^2$.	1. *Why?*
2. $h^2 = a^2 - (c - b')^2$ (in I), or $h^2 = a^2 - (b' - c)^2$ (in II).	2. *Why?*

Complete the proof by equating the values of h^2 found in 1 and 2.

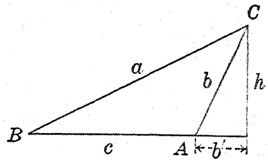
Ex. 1. If $b = 10$ in., $c = 21$ in., and $h_c = 8$ in., find a.

Ex. 2. If $b = 13$ in., $c = 14$ in., and $b' = 5$ in., find a.

***Ex. 3.** If $b = 43$ in., $c = 61$ in., and $p^c_b = 11$ in., find a.

***Ex. 4.** If $a = 48$ in., $b = 29$ in., and $p^b_a = 20$ in., find c.

329. A. **Theorem.** *In any obtuse triangle, the square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides by the projection of the other side on it.* (Prop. 12)



Given: $\triangle ABC$ with obtuse $\angle A$, side a opposite $\angle A$, b and c the other sides, b' the projection of b on c, h the altitude on c.

To prove: $a^2 = b^2 + c^2 + 2\,b'c$.

Proof: Proceed as in § 328.

330. A. Corollary. *If* $a^2 < b^2 + c^2$, $\angle A$ *is an acute angle; if* $a^2 = b^2 + c^2$, $\angle A$ *is a right angle; and if* $a^2 > b^2 + c^2$, $\angle A$ *is an obtuse angle.*

331. B. If a, b, and c are sides of a triangle and A is an acute angle, we have the formulas: $a^2 = b^2 + c^2 - 2\,cp_c^b$, or $a^2 = b^2 + c^2 - 2\,bp_b^c$.

If A is an obtuse angle, we have the formulas:

$$a^2 = b^2 + c^2 + 2\,cp_c^b, \text{ or } a^2 = b^2 + c^2 + 2\,bp_b^c.$$

EXERCISES

The following exercises can be solved by the formulas in §§ 330, 331.

1. The sides of a triangle are 8 in., 9 in., and 12 in. What kind of angle is the largest angle of the triangle?

2. The sides of a triangle are 12 yd., 16 yd., and 20 yd. What kind of angle is the largest angle of the triangle?

If $a = 13$ *in.*, $b = 15$ *in.*, $c = 14$ *in.*:

3. What kind of angle is $\angle A$?

4. Find p_c^b; p_c^a.

5. Find h_c.

If $a = 17$ in., $b = 10$ in., $c = 21$ in.:

6. What kind of angle is $\angle C$?
7. Find p^b_c; p^a_c.
8. Find h_c.

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9. If $a = 35$ in., $b = 29$ in., and $p^a_c = 28$ in., find c.
10. If $a = 43$ in., $b = 68$ in., and $p^b_a = 32$ in., find c.
11. If $a = 25$ in., $b = 26$ in., and $p^b_c = 10$ in., find c.
12. If $a = 25$ in., $c = 25$ in., and $p_c = 7$ in., find b.
13. If $a = 18$ in., $b = 12$ in., and $\angle C = 60°$, find p^c_b.
14. If $a = 8$ in., $b = 6$ in., and $\angle C = 30°$, find h_b.
15. If $c = 21$ in., $\angle B = 45°$, and $p^a_c = 9$ in., find h_c; find p^b_c.
16. If $a = 20$ in., $b = 12$ in., and $\angle C = 120°$, find c.
17. Prove the principle used by the Egyptians, that a triangle whose sides are proportional to 3, 4, and 5 is a right triangle.

SUGGESTION. — Let $3n$, $4n$, and $5n$ be the three sides.

18. The legs of a right triangle are 9 in. and 12 in., respectively. Find the lengths of their projections upon the hypotenuse.
19. Prove that if A is an acute angle, $p^b_c = \dfrac{b^2 + c^2 - a^2}{2c}$.
20. Prove that if $\angle A$ is obtuse, $p^b = \dfrac{a^2 - b^2 - c^2}{2c}$.
21. Write the formulas for b^2 if $\angle B$ is acute; if $\angle B$ is obtuse.
22. Write the formulas for c^2 if $\angle C$ is acute; obtuse.
23. Can $a^2 = b^2 + c^2 + 2cp^b_c$ be written $a^2 = b^2 + c^2 + 2bp^c_b$?

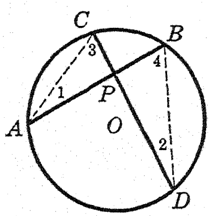
*24. In $\triangle ABC$, $\angle A$ is acute and $a^2 = b^2 + c^2 - 2bp^c_b$. If angle A increases in size, what happens to p^c_b? Draw a figure to determine. If $\angle A$ becomes 90°, how large is p^c_b?

332. A. **Continuity.** When the projection of side b on c lies on side c, we may consider it *positive*; but when the projection lies on c produced, it is *negative*. Then §§ 328 and 329 may be considered as special cases of the Pythagorean theorem and we always have

$$a^2 = b^2 + c^2 - 2cp^b_c.$$

PROPOSITION 13. THEOREM

333. *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*



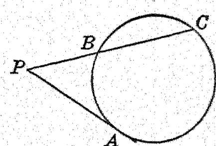
Given: Circle O, with chords AB and CD, intersecting at P.

To prove: $AP \cdot PB = CP \cdot PD$.

Plan: Show that $\triangle APC \sim \triangle BPD$.

Proof:

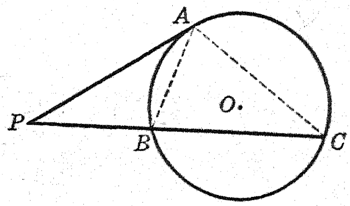
STATEMENTS	REASONS
1. Draw AC and BD. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.	1. § 241.
2. $\angle CPA = \angle DPB$.	2. § 46.
3. $\triangle ACP \sim \triangle BDP$.	3. § 305.
4. $AP \cdot PB = CP \cdot PD$.	4. *Why?*



334. Secants and tangents. In the figure, PA is tangent to the circle at A and PBC is a secant. The segment PA is called the **length of the tangent,** the length PC the **whole secant** and PB the **external segment.** Can you prove that PA is the mean proportional between PB and PC? Draw BA and CA and prove $\triangle PBA \cong \triangle PAC$.

Proposition 14. Theorem

335. *If, from a point outside a circle, a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.*



Given: Circle O, with PBC a secant and PA tangent at A.

To prove: $PC : PA = PA : PB$.

Plan: Prove that $\triangle PAC$ and PAB are similar.

Proof: Write in full. See Ex. 10, § 317.

336. Corollary. *If, from an external point, secants are drawn to a circle, the product of each secant by its external segment is a constant.*

EXERCISES

In the figure for § 335:

1. If $AP = 30$ in., and $PC = 45$ in., find PB.
2. If $AP = 16$ in., and $PC = 64$ in., find PB and BC.
3. If $PB = 8$ in., and $BC = 10$ in., find PA.
4. If $AP = 8$ in., and $PC = 20$ in., find PB.
5. If $AP = 12$ in., and $BC = 10$ in., find PC.

In the figure for § 333:

6. If $AP = 6$ in., $PB = 10$ in., and $PC = 5$ in., find PD.
7. If $AB = 13$ in., $PB = 9$ in., and $PD = 6$ in., find CD.

8. If $CD = 11$ in., $CP = 3$ in., and $AP = 4$ in., find AB.

9. If $AB = 16$ in., $CD = 20$ in., and $PD = 4$ in., find AP.

10. If $AB = 17$ in., $CD = 13$ in., and $AP = 3$ in., find CP.

11. Two secants are drawn from an external point to a circle. One secant and its external segment are 14 in. and 6 in., respectively. If the other secant is 12 in., find its external segment.

12. A tangent and a secant are drawn from a point to a circle, the secant passing through the center. If the tangent is 10 in. long and the external segment of the secant 4 in. long, find the radius.

13. In a circle whose radius is 12 in., a chord 18 in. long is drawn through a point 8 in. from the center. Find the segments into which the chord is divided at the point.

*14. (Figure § 333) If $AB = 8$ in., $CD = 12$ in., and $AP = 5$ in., find CP.

*15. (Figure § 335) If $AP = 12$ in. and $BC = 12$ in., find PC.

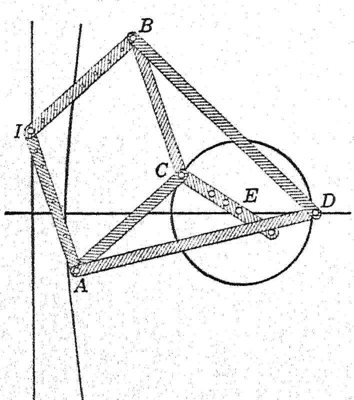


FIG. 1

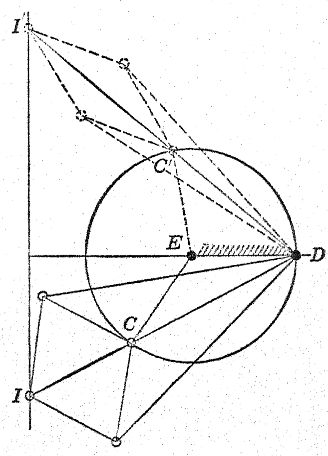
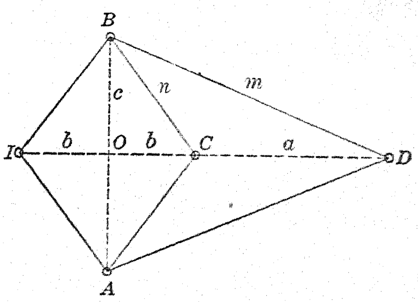


FIG. 2

NOTE. — In 1864 M. Peaucellier, an officer in the French army, invented the *linkage* (see § 264) shown above in Fig. 1. Points D and E are fastened to a base so that, as point C traces a circle, point I traces a straight line perpendicular to DE produced.

*16. The illustration below shows *Peaucellier's cell.* $DB = DA = m$, and $BC = BI = IA = AC = n$. Prove that points I, C, and D lie in a straight line.

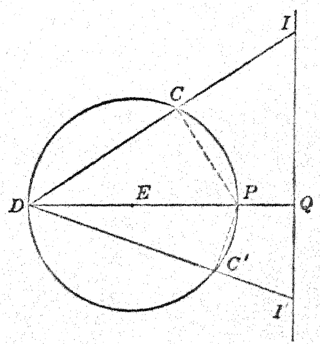
HINT. — How is ID related to BA? See § 87.



*17. Show that, no matter what the position of the links in Peaucellier's cell, $\overline{DC} \times \overline{DI}$ is a constant.

HINT. — By applying § 326 to triangles DOB and COB, show that $m^2 - n^2 = a(a + 2b)$; that is, $m^2 - n^2 = \overline{DC} \times \overline{DI}$.

*18. The diameter DP of a circle is produced to any point Q and II' is drawn perpendicular to DQ. Secants DCI and $DC'I'$ are drawn through D and cut the circle at C and C' and the perpendicular at I and I'. Prove that $\overline{DC} \times \overline{DI} = \overline{DC'} \times \overline{DI'} = \overline{DP} \times \overline{DQ}$. That is, prove that the product of the segments made by the circle and II' on any secant drawn from point D is a constant.



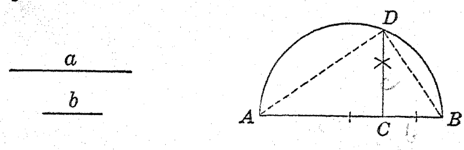
HINT. — Draw CP and compare similar right triangles DQI and DCP.

*19. Referring to Fig. 2 on page 320 and to Ex. 16–18, show why, as point C traces the circle with center E, point I traces the straight line II' perpendicular to DE.

NOTE. — When the bar CE (Fig. 1, page 320) is lengthened, I traces an arc of a circle.

Construction XX

337. *To construct the mean proportional between two given line segments.*



Given: Segments a and b.

Required: Construct the mean proportional between a and b.

Construction: Write the construction and proof. See § 324.

Ex. 1. Construct a segment equal to $\sqrt{6}$ in.

HINT. — If x is the required segment, $x^2 = 6$. Therefore $2 : x = x : 3$.

Ex. 2. Construct a segment equal to $\sqrt{3}$ in.

Ex. 3. Construct a segment $a\sqrt{3}$ in. long, if a is a segment of given length.

Ex. 4. If $xa = b^2$, construct x. (Recall § 289–*6*.)

Ex. 5. If a quadrilateral is inscribed in a circle, the product of the segments of one diagonal equals the product of the segments of the other diagonal.

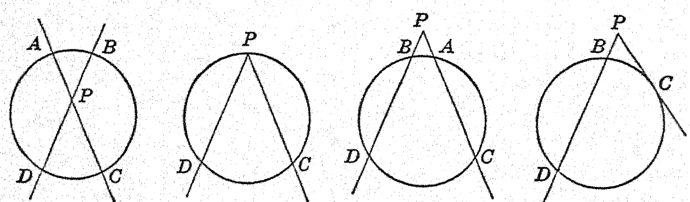


FIG. 1 FIG. 2 FIG. 3 FIG. 4

338. A. **Continuity.** The theorems in §§ 333, 335, and 336 can be stated as a single theorem by considering that the point P divides the lines AC and BD *internally* in Fig. 1 and *externally* in Fig. 3. In each case the segments of AC are PA and PC and the segments of BD

are PB and PD. In each case $\overline{PA} \cdot \overline{PC} = \overline{PB} \cdot \overline{PD}$. In Fig. 2, PA and PB are each zero; in Fig. 4, $PA = PC$. Draw the figure for two tangents.

339. Summary of the Work of Unit Six.

I. ***You can prove two triangles are similar by proving that:***

1. *They are mutually equiangular.*
2. *They have an angle of one equal to an angle of the other and the including sides proportional.*
3. *They have their sides, respectively, proportional.*

II. ***You can prove that four segments are in proportion by proving that:***

1. *They are corresponding segments intercepted by parallel lines.*
2. *They are segments intercepted on two sides of a triangle by a line parallel to the third side.*
3. *They are corresponding sides of similar triangles.*

III. ***You can prove that the product of two segments is equal to the product of two other segments by proving that:***

1. *One pair is the means and the other pair the extremes in a proportion.*
2. *They are segments of two intersecting chords in a circle.*
3. *They are formed by two secants drawn from an external point to a circle.*

IV. ***You can prove that one segment is the mean proportional between two others by using the theorems about:***

1. *A tangent and a secant drawn from an external point.*
2. *The altitude drawn from the vertex of a right triangle to the hypotenuse.*

V. ***Constructions:***

1. *To construct the fourth proportional to three given line segments.*
2. *To divide a segment into parts proportional to any number of given segments.*
3. *To construct the mean proportional between two given line segments.*

REVIEW OF UNIT SIX

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the tests. The references given are those most closely related to the exercise.

1. What is a ratio? A proportion? § 285, § 286.

2. Define antecedent; consequent; means; extremes. §§ 287, 288, Ex. 6.

3. Is the ratio of similitude of two similar polygons the ratio of any two sides? § 303.

4. Is the fourth proportional the fourth term in any proportion? § 290.

5. Is the third proportional the third term in any proportion? § 290.

6. What is the mean proportional? § 290.

7. What two conditions are necessary for the similarity of two polygons? § 302.

8. Write the proportion $\frac{a}{b} = \frac{c}{d}$ by alternation; by inversion; by addition; by subtraction. § 289.

9. If $xy = zw$, write four proportions involving x, y, z, and w. § 289.

10. If two polygons have their corresponding sides proportional, are they similar? § 302.

11. If two triangles have their corresponding sides proportional, are they similar? § 314.

12. If two polygons are mutually equiangular, are they similar? § 302.

13. If two triangles are mutually equiangular, are they similar? § 305.

14. Must similar polygons have the same number of sides? Of angles? § 302.

15. How can you find geometrically $\sqrt{7}$? § 337.

16. How can you construct a segment x, if $x^2 = ab$? If $x = \frac{a^2}{b}$? If $ax = b^2$? §§ 297, 337.

17. How do you divide a segment into parts proportional to m, n, and p? § 299.

18. Is it correct to say: "Segment AB is proportional to segment CD"? § 286.

19. If $a : b = c : d$, what can you say about ad? § 289.

20. If $a : b = c : b$, what can you say about $c : a$? § 289.

21. If $a : b = c : d$ and $a : b = c : e$, what can you say about e? § 289.

Complete the following:

22. Corresponding altitudes of similar triangles . . . § 306.

23. Two triangles are similar if an angle of one . . . § 312.

24. The bisector of an interior angle of a triangle divides the opposite side . . . § 317.

25. If a perpendicular is drawn from the vertex of a right triangle to the hypotenuse:

(*a*) The perpendicular is the mean proportional . . . § 323.
(*b*) Either side is the mean proportional . . . § 323.

26. In any right triangle the square . . . § 326.

27. If two chords intersect in a circle, . . . § 333.

28. If from a point outside a circle a secant and . . . § 335.

29. The bisector of an exterior angle of a triangle divides the opposite side ... § 319.

30. If P divides a segment AB externally, the two parts are ... and ... § 318.

31. A. The ratio of the *side adjacent* to the *hypotenuse* is called the ... § 307.

32. A. The tangent is the ratio of ... § 307.

33. A. If, in a right triangle, you are given an acute angle and the *side adjacent*, the *side opposite* can be found by using the ... § 307.

34. A. In any triangle, the square of a side opposite an acute angle ... § 328.

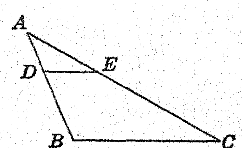
35. A. In any obtuse triangle, the square of a side opposite an obtuse angle ... § 329.

36. A. Points P and Q divide segment AB harmonically. How can you find the length of the external segments if you are given the length of the internal segments? § 320.

37. Tell how to find the diameter of a circle if you know the length of the tangent from a point P to the circle and the distance of P from the nearer arc of the circle. § 335.

38. Tell how to find the length of a chord if you know the length of the diameter and the distance from the center of the circle to the chord. § 333.

NUMERICAL EXERCISES



1. In $\triangle ABC$, $AB = 3$ in., $BC = 4$ in., $CA = 6$ in. If $DE \parallel BC$ and $AD = 1$ in., find AE and EC.

2. If $AD = 4$ in., $DB = 3$ in., $AE = 6$ in., and $DE \parallel BC$, how long is EC?

3. If $DE \parallel BC$, $AB = 8$ in., $DB = 3$ in., $AC = 10$ in., and $BC = 12$ in., find AE and DE.

4. If $DE \parallel BC$, $BD = 12$ in., $AD = 18$ in., $CB = 25$ in., and $AC = 50$ in., find DE and EC.

5. If $DE \parallel BC$, $AB = 4$ in., $AE = 12$ in., and $AD = 2\ DB$, find AD, DB, and EC.

6. If $AB = 8$ in, $AD = 2$ in., $AE = 3$ in., and $AC = 12$ in., is $DE \parallel BC$?

In Ex. 7–11, given $\angle C$ in $\triangle ABC$ a right angle, with $CD \perp AB$. (See the Appendix for the solution of quadratic equations.)

7. If $AC = 6$ in. and $AB = 18$ in., find AD, BC, and CD.

8. If $BC = 12$ in. and $AD = 7$ in., find AB.

SUGGESTION. — $BC^2 = AB \times BD$. If $BD = x$,

$$\text{Then } x(x+7) = 144$$
$$x^2 + 7x - 144 = 0$$
$$(x+16)(x-9) = 0$$
$$x = 9$$

9. If $BC = 12$ in. and $AD = 18$ in., find AC, BD, and AB.

***10.** If $BC = 9$ in. and $AD = 6$ in., find AB.

***11.** If $BD = 12$ in. and $AC = 20$ in., find AB.

12. The hypotenuse of an isosceles right triangle is 12 in. Find the other sides.

13. The side of an equilateral triangle is 20 in. Find the altitude.

14. If one acute angle of a right triangle is 45° and a leg is a, find the hypotenuse.

15. If one acute angle of a right triangle is 30° and the leg opposite that angle is 5 in., find the hypotenuse and the other leg.

16. The altitude of an equilateral triangle is 20 in. Find the side.

17. In $\triangle ABC$, $\angle B = 30°$, $\angle C = 45°$, and $AB = 8$ in. Find AC and BC.

HINT. — Draw $AD \perp BC$.

18. In $\triangle ABC$, $\angle C = 135°$, $\angle B = 30°$, and $AB = 10$ in. Find AC and BC.

HINT. — Draw $AD \perp BC$ produced.

19. In $\triangle ABC$, $\angle C = 120°$, $\angle B$ is 45°, and $AB = 20$ in. Find AC and BC.

20. Chords AB and CD intersect at E within a circle. If AB is 24 in., CE is 16 in., and ED is 5 in., find the segments of AB.

21. If chord AB is 28 in., CE is 8 in., and ED is 12 in., find AE.

22. The sides of a triangle are 12 in., 15 in., and 18 in. Find the segments of the 18 in. side made by the bisector of the opposite angle.

23. In Ex. 22 find the segments of side 15 in. made by the bisector of the opposite angle.

24. A map is drawn to the scale of 1 in. to 10 mi. How far apart are two places that are $3\frac{1}{8}$ in. apart on the map?

25. In $\triangle ABC$, $BC = 15$ ft. and altitude $AD = 7$ ft. In similar $\triangle A'B'C'$ side $B'C'$ corresponding to BC is 50 yd. Find altitude $A'D'$.

***26.** In trapezoid $ABCD$, $AB \parallel CD$. If $AB = 16$ ft., $BC = 10$ ft., $CD = 12$ ft., $DA = 15$ ft., and if AD and BC meet at E, find the perimeter of $\triangle ECD$.

27. The bases of a trapezoid are 12 in. and 15 in.; its altitude is 8 in. If the non-parallel sides are produced to meet, what is the altitude of each triangle thus formed?

28. A. The angle of elevation of a house at a distance of 300 feet is 20°. Find the height of the house.

29. A. From a lighthouse the angle of depression of a boat was 18°. If the lighthouse was 125 feet high, how far away was the boat?

30. When a ship sails for 3 hours at an average speed of 18 miles per hour on a course 30° west of south, how far west of its starting point is it?

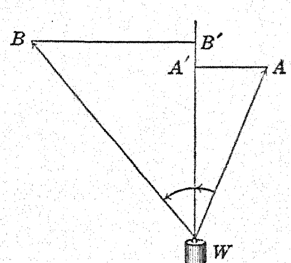
31. A. Two forces, one of 80 lb. and the other of 96 lb., are exerted on an object at A at an angle of 30°. Find the value of the resultant force and the angle it makes with the given force.

HINT. — See Ex. 1, page **141**.

32. A. An airplane is flying on a compass course due east at the rate of 90 miles per hour. If a north wind causes the plane to drift south at the rate of 12 miles per hour, find the direction in which the plane is moving and the distance it travels in 10 hours.

33. A. If the average inclination of a river bed is 6° 10′ with the horizontal, how far does it descend in 1 mile of its course?

***34.** A. The Washington Monument is 555 ft. high. From points due west of the Monument, two observers note that its angles of elevation are 25° and 42° 20′, respectively. How far apart are the observers?



***35.** A. Two men are lifting a stone by means of ropes. One man pulls 90 lb. in a direction 23° from the vertical, and the other man pulls 105 lb. in a direction 40° from the vertical. Find the weight of the stone.

HINT. — The total weight of the stone is $WA' + WB'$.

CONSTRUCTIONS

1. Construct geometrically $\sqrt{5}$.

2. Construct the third proportional to two segments 2 in. and 3 in.

3. Construct the fourth proportional to the segments 1 in., $1\frac{1}{2}$ in., and 2 in.

4. Divide a segment $4\frac{5}{8}$ in. long into parts proportional to 1, 3, and 5.

*5. Inscribe in a given circle a triangle similar to a given triangle.

HINT. — Circumscribe a circle about the given triangle and copy in the given circle the central angles formed by each side.

*6. Construct a circle which shall pass through two given points and be tangent to a given line.

HINT. — Let A and B be the given points and let AB intersect the given line l in P. Then if X is the supposed point of tangency of l, $PA \times PB = \overline{PX}^2$, by § 335.

*7. Construct a circle through a given point and tangent to two given lines.

HINT. — Draw a line through the given point perpendicular to the bisector of the angle formed by the given lines. On this line produced, locate a second point the same distance from the bisector as that of the given point. Then use Ex. 6.

GENERAL EXERCISES

1. Prove that corresponding angle bisectors of two similar triangles have the same ratio as any two corresponding medians.

2. In $\triangle ABC$, AD is drawn to a point in BC so that $\angle CAD = \angle B$. Prove $\triangle ACD \sim \triangle ABC$.

3. Prove that the altitudes of two similar trapezoids have the same ratio as any two corresponding sides.

4. Prove that two corresponding diagonals of two similar hexagons have the same ratio as any two corresponding sides.

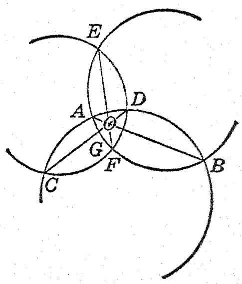
5. Prove the converse of § 317.

HINT. — Use the indirect method.

6. Prove that the product of the segments of all chords drawn through a point within a circle is a constant.

7. Prove that if, from an external point, any number of secants are drawn to a circle, the product of any secant and its external segment is a constant.

***8.** If three circles intersect one another, the three common chords all pass through the same point.



SUGGESTION. — Let the chords AB and CD intersect at O. Draw EO and produce it. Suppose that EO produced meets arc EAB again at F and arc EDC at G. Prove that $OF = OG$, and hence that F and G coincide.

***9.** If two circles are tangent externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circles are proportional.

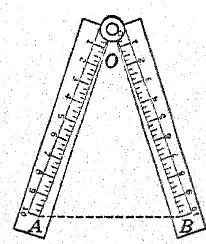
***10.** If two circles are tangent externally their common external tangent is the mean proportional between their diameters.

***11.** Prove the converse of § 319.

PRACTICAL APPLICATIONS

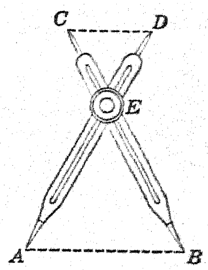
(OPTIONAL)

1. The instrument shown in the figure is called a *sector*. By means of it various constructions and measurements can be made. Thus, to bisect a segment, open the sector until the transverse distance from A to B on the scales OA and OB equals the given segment. Then the distance between the midpoints of OA and OB is equal to one half of the segment. Give the proof.



2. Show how the sector may be used to divide a line segment into five equal parts, by opening them until the transverse distance between the fifth divisions on the scales equals the given segment. Give the proof.

3. Show by use of the sector how to divide a given segment into nine equal parts.

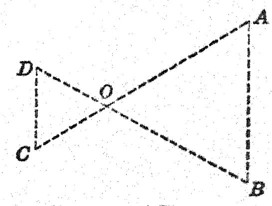


4. The instrument shown in the drawing is a pair of *proportional compasses*. The lengths AE, BE, ED, and EC are adjusted proportionally by means of the screw at E. Prove that $\frac{AB}{CD} = \frac{AE}{ED}$.

If $AE = 8$ in. and $ED = 2$ in., the distance between A and B is how many times the distance between C and D?

5. The following method may be used for estimating the distance from the observer to an inaccessible object:

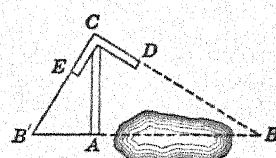
With the left eye closed, the finger is pointed, at arm's length (at O), toward the object A. Then without moving the finger, the right eye is closed and the left eye opened, when the object appears to have moved to B. The distance AB through which it appears to have moved, being transverse to the line of sight, is estimated. The distance from the finger O to the object is approximately 10 times the distance AB.



Show that, if the distance CO from the eye to the outstretched finger of the average person is approximately 10 times the distance CD between the eyes, $OA = 10\ AB$.

6. The French liner *Normandie* (1935) is 981 ft. long. When observed at sea by the method of Ex. 5, it appeared to move through a distance of four ship lengths. How far away was it? How many miles?

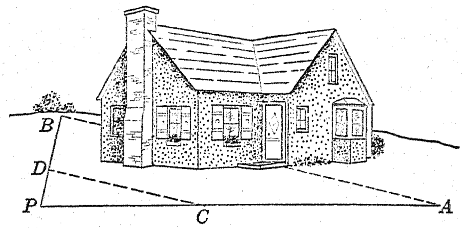
7. Several centuries ago, before modern instruments were invented, a method used for determining the distance from A to an inaccessible point B was as follows: Upon a vertical staff AC was placed an instrument resembling a carpenter's square. The blade CD was pointed toward B, and at the same time the point B' on the ground at which the blade CE pointed was marked. $B'A$ and AC were measured. Then AB was computed by the proportion $B'A : AC = AC : AB$.



Prove that this proportion is true.

8. In Ex. 7, if $AB' = 6$ in. and $AC = 62$ in., find AB.

9. The distance between two accessible points A and B which are separated by an obstacle may be measured as follows: From a convenient point P the distances PA and PB are measured. Then in these lines points C and D, respectively, are located so that $PC : PA = PD : PB$. Finally, CD is measured.

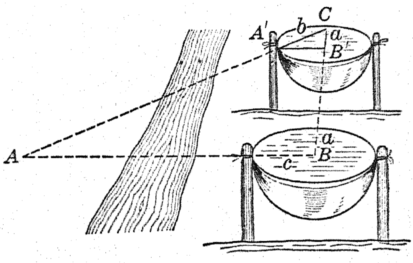


Prove the proportion by means of which AB may now be computed.

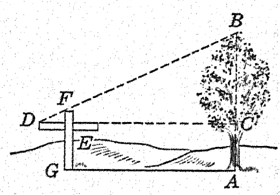
If $PA = 240$ ft., $PC = 40$ ft., $CD = 20$ ft., find AB.

Use the method explained here to find the distance between two inaccessible points.

10. Before modern instruments were invented, an inaccessible distance AB was measured by use of *drum heads.* On a drum head placed at B a line a was drawn toward C and a line c toward an accessible point A. BC was then measured, and the drum head removed to C and placed with a in the direction BC as indicated. Then a third line b was drawn toward A. Show how it was possible from these measurements to compute AB. If $BC = 200$ yd., $B'C = 12$ in., and $c = 16$ in., compute AB.



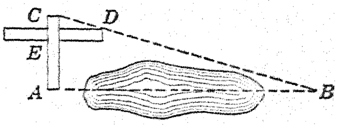
11. The *cross-staff* may be used to find the height AB of an object as follows: The horizontal cross-bar DE is raised or lowered on the staff FG until D, F, and B fall in a straight line. Then DE, EF, GE, and GA are measured.



Explain how AB may be computed.

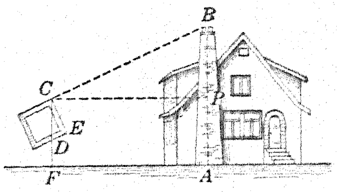
If $DE = 18$ in., $EF = 6$ in., $GE = 5$ ft., and $GA = 60$ ft., find AB.

12. Explain how the cross-staff may be used to obtain the horizontal distance from a point A to an inaccessible point B.



13. In Ex. 12, if AC is 5 ft., EC is $2\frac{1}{4}$ in., and ED is 18 in., find AB.

14. The *geometric square* was used in practical measurements before modern engineering instruments were invented. It consists of a square frame, along two adjacent edges of which is marked a scale, and from the opposite corner of which a plumb line is suspended. A pair of sights on another edge aid in pointing the instrument.

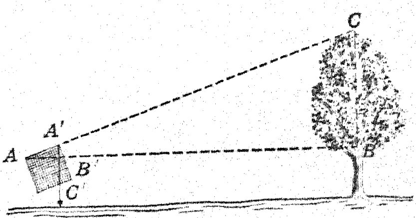


When the height AB of an object is to be found, the square is held in a vertical plane and the edge bearing the sights is pointed toward B. The point D where the plumb line then crosses the scale is noted. PB is computed by proportion.

Prove that $CE : CP = DE : PB$.

15. If CE is 12 in., DE is 8 in., FA is 100 ft., and CF is 5 ft., find AB.

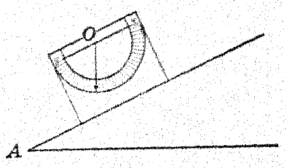
16. A modern form of the geometric square can very easily be made by tacking a piece of graph paper on a board and attaching sights and a plumb bob.



Show that $\triangle A'B'C' \sim \triangle ABC$ and that the length of BC can be computed from $B'C'$

Such an instrument is called a *hypsometer*.

17. If A is 5 ft. from the ground, AB is 60 ft., $A'B'$ is 50 and $B'C'$ is 20 find the height of the tree.

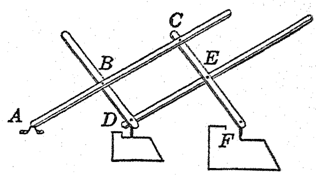


18. If a large protractor is made and tacked to a board with a pointer swinging freely at O, the angle A of any slope can be read from the protractor.

Show that the angular reading at O is equal to $\angle A$.

This instrument is called a *clinometer*.

19. A *pantograph* is an instrument for drawing a figure similar to a given figure, and is useful for enlarging or reducing maps and drawings. It consists of four bars, parallel in pairs and jointed at B, C, D, and E. A turns on a fixed pivot, and pencils are carried at D and F. BD and DE are so adjusted as to form a parallelogram $BCED$ and such that any required ratio $AB : AC$ is equal to $CE : CF$.

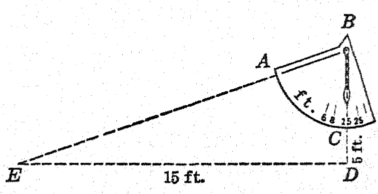


Show that A, D, and F are always in a straight line.

Hint. — Prove that $\angle DAB = \angle FAC$.

20. Show that the ratio $AD : AF$ remains constant and equal to $AB : AC$ so that if the point D traces a given figure, the pencil F will trace a similar figure.

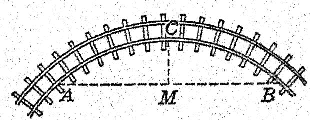
21. A. The instrument shown in the drawing is called a *telemeter*. It is used for estimating distances. AB is pointed toward the object E to which the distance is to be found. The plumb line points on the scale to the number of feet that the object E is distant. The instrument is made to be held at a height BD of 5 ft. above the ground.



In making the scale of feet on the telemeter, the 15-ft. mark of the scale must be placed at a point C so that $\angle ABC$ is how many degrees?

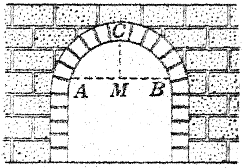
Compute in the same way the number of degrees in $\angle ABC$ when C is at the 6-ft. mark. When C is at the 25-ft. mark. (See sections on numerical trigonometry, §§ 307–310.)

22. A railroad surveyor who wished to find the radius of the railroad curve ACB, measured the chord AB and the distance CM from the middle point of the arc to the middle point of the chord. Show how he was then able to compute the radius.



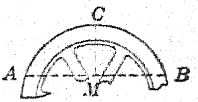
If AB is 100 ft. and CM is 2 ft., compute the radius of the curve.

23. A window whose width AB is 5 ft. is to be surmounted by a circular stone arch of which the rise CM is 20 in. Find the radius of the circle at which the stone for the arch must be cut.



24. In a bridge, a circular arch 18 ft. high is to span a stream 72 ft. wide. What is the radius of the circle at which the stones of this arch must be cut?

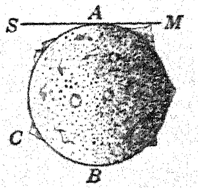
25. This is a piece of broken wheel. If AB = 16 in. and CM = 4 in., find the diameter of the wheel.



26. Three stakes are set in a canal 2 mi. long, one at each end and one in the middle, and all project the same distance above the water. By use of a leveling instrument the middle stake is found to be 8 in. higher than the others. From these facts compute the diameter of the earth.

27. A bridge spans a stream 80 ft. wide, and the stone arch of the bridge is 25 ft. above the water at the center. Find the radius of the arch.

28. Galileo, who lived about 1600, measured the heights of the mountains on the moon as follows: ACB was the illuminated half of the moon just as the peak of the mountain M caught the beam SM of the rising or setting sun. He measured the distance AM from the half-moon's straight edge AB to the mountain peak M. Then by using the known diameter of the moon, show how he was able to compute the height of the mountain.

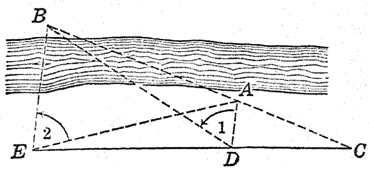


29. In the figure of Ex. 28 the known diameter AB of the moon is about 2160 miles. If the distance AM is found to be 100 mi. how high is the mountain M?

30. Assuming the diameter of the earth to be 8000 mi., how far can one see on the surface of the earth from the top of a mountain 2 mi. high which rises out of a level plain?

31. What is the greatest distance on the surface of the earth that a man can see from an airplane if he is one mile high?

32. A surveyor proceeded as follows to find the approximate distance from a point A to an inaccessible point B: He walked from A 100 paces to C in a line with A and B. Then he walked 100 paces to a convenient point D, and noted $\angle 1$. He continued in line CD until he reached a point E from which $\angle 2 = \angle 1$. Show that $DE = AB$.



Suggestion. — It can be proved that $DE = AB$ if it is first proved that a circle may be drawn through the four points A, B, E, and D. See § 280. Then use § 335.

PRACTICE TESTS

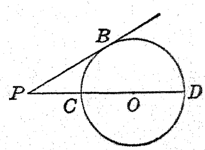
These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT SIX

TEST ONE

Numerical Exercises

1. PB is a tangent and CD is a diameter. If PB is 12 in. and PC is 8 in., what is the radius of the circle? § 335.

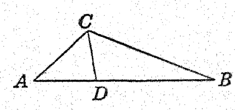


2. In $\triangle ABC$, AB is 18 in., BC is 12 in., and AC is 14 in. A line PQ parallel to BC is 10 in. Find AP if point P is on AB. § 302.

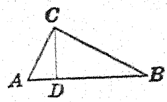
3. Find the fourth proportional to 3, 36, and 2. § 290.

4. Chords MN and PR intersect at C. If PC is 4 in., CR is 12 in., and CN is 6 in., find MN. § 333.

5. If CD bisects $\angle C$, AC is 6 in., BC is 14 in., and AB is 15 in., how long is DB? § 317.

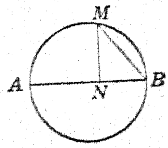


6. How high is a tower which casts a shadow 110 ft. long when a vertical pole 6 ft. high casts a shadow 8 ft. long? § 302.



7. In $\triangle ABC$, $\angle C$ is a right angle and $CD \perp AB$. If AB is 25 in. and AD is 5 in., how long is CD? § 323.

8. The perimeters of two equilateral triangles are 36 in. and 24 in., respectively. What is the ratio of their altitudes? § 306.



9. If diameter AB is 18 in. and MN is perpendicular to AB, find MB if NB is 8 in. § 324.

10. In the figure of Ex. 7, if BC is 35 in. and AB is 37 in., how long is AC? § 326.

11. Find the length of the longest and the shortest chord that can be drawn through a point 5 in. from the center of a circle whose radius is 13 in. § 333.

12. In $\triangle ABC$, $AC = 6$ in., $BC = 10$ in., $AB = 10$ in., and $PQ \parallel AB$. P and Q are points on AC and BC, respectively. If BQ is 2 in., how long is PQ? § 302.

TEST TWO

True-False Statements

If a statement is always true, mark it so. If it is not always true, replace each word in italics by a word which will make it a true statement.

1. Two isosceles right triangles are *similar*. § 312.

2. The corresponding angle bisectors of two triangles have the same ratio as any two corresponding sides if the sides of the triangles are *perpendicular* each to each. § 306, Ex. 13.

3. If two polygons are similar, their corresponding sides are *proportional*. § 302.

4. Corresponding *altitudes* of similar triangles have the same ratio as corresponding sides. § 306.

5. If two polygons are *similar*, their corresponding angles are equal. § 302.

6. In similar triangles, if we know the ratio of two corresponding sides, we also know the ratio of corresponding *angle bisectors*. § 306, Ex. 13.

7. A *third* proportional to two numbers C and D, is Y in the following: $C : Y = Y : D$. § 290.

8. If two polygons are similar, they are *equiangular.* § 302.

9. Two triangles are *congruent* if the sides of one are respectively perpendicular to the sides of the other. § 305.

10. All *rectangles* are similar. § 302.

11. Similar triangles are *never* congruent. § 302.

12. Two polygons may be mutually equiangular without being *similar.* § 302.

TEST THREE

Multiple-Choice Statements

From the expressions printed in italics select that one which best completes the statement.

1. The median on the hypotenuse of a right triangle divides the triangle into two triangles which are *congruent, similar, isosceles, equilateral.* § 159.

2. A proportion is *a ratio, a statement about similar figures, an equality between equal ratios.* § 286.

3. In the proportion $a : b = b : c$, b is called the *third, fourth, mean* proportional. § 290.

4. If a line divides two sides of a triangle proportionally, it is *parallel to, proportional to, equal to half* the third side. § 300.

5. Mutually equiangular polygons *are similar, have each angle of one, respectively, equal to the corresponding angles of the other taken in order, have all their angles equal.* § 301.

6. Two triangles are similar if their corresponding *angles, sides* are proportional. § 314.

7. The bisector of an interior angle of a triangle divides the opposite side into segments which are *equal, proportional, similar* to the adjacent sides. § 317.

8. In any right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, this perpendicular is the mean proportional between *the two legs, the hypotenuse and an adjacent side, the segments of the hypotenuse.* § 323.

9. If from any point on a circle a perpendicular is drawn to a diameter, *the segment connecting the point with the extremity of the diameter, the perpendicular,* is the mean proportional between the diameter and its adjacent segment. § 324.

10. If two chords intersect in a circle *they bisect each other, their products are proportional, the product of the segments of the one is equal to the product of the segments of the other.* § 333.

11. If two secants are drawn from an external point to a circle *the product of one secant by its external segment is equal to the product of the other secant by its external segment, the product of the segments of the one is equal to the product of the segments of the other, they divide the circle into equal arcs.* § 336.

12. In showing that the ratio between two segments a and b is the same as the ratio between two other segments c and d, *the same unit must be used in measuring all four segments, different units may be used, one unit may be used to measure a and b and a different one to measure c and d.* § 285.

CUMULATIVE TESTS ON THE FIRST SIX UNITS

TEST FOUR

Numerical Exercises

1. The ratio of each interior angle of a regular polygon to each exterior angle is 3 to 1. How many sides has the polygon? § 134.

2. AB is the diameter of a circle whose radius is $14\frac{1}{2}$ in. If chord AC is 20 in., find chord BC. §§ 243, 326.

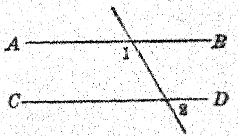
3. In $\triangle ABC$, AB is 25 in., BC is 15 in., and AC is 20 in. From P on BC a line PQ is drawn parallel to AC. Find BP if PQ is 16 in. § 302.

4. In a circle whose diameter is 40 in., a chord is drawn 16 in. from the center. What is the product of the segments of the chord? § 333.

5. If the vertex angle of an isosceles triangle is three times the sum of the base angles, how many degrees are there in each angle of the triangle? § 123.

6. How many sides has a polygon the sum of whose interior angles is eight right angles? § 133.

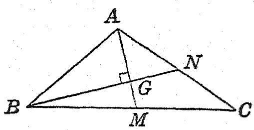
7. $AB \parallel CD$; $\angle 1 = 115°$. How many degrees are there in $\angle 2$? § 113.



8. How many degrees are in the angle formed by the bisectors of two angles of a triangle, 45° and 75° respectively? § 123.

9. Given: Right triangle ABC with right angle at C; $\angle A = 30°$. $BC = 7$ in. Find the length of AC. §§ 160, 326.

10. The hypotenuse AB of right triangle ABC is 28 in. How long is the median from C to AB? § 159.



11. AM and BN are medians of $\triangle ABC$. They intersect at G and are perpendicular. If $AM = 13\frac{1}{2}$ in. and $BN = 18$ in., how long is AB? §§ 278, 326.

12. Given $\triangle ABC$ with $AB = 8$ in., $AC = 9$ in., and $BC = 10$ in. Find the segment BD made on BC by the bisector of the exterior angle at A. § 319.

TEST FIVE

True-False Statements

If a statement is always true, mark it so. If it is not always true, replace each word in italics by a word which will make it a true statement.

1. The *perpendicular* to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. § 323.

2. In a right triangle, if one acute angle is 30°, the *longer* leg and the median to the hypotenuse are equal in length. §§ 159, 160.

3. If an isosceles triangle is obtuse, the base is the *shortest* side. § 172.

4. An *equiangular* polygon is a polygon all of whose angles are equal. § 135.

5. *Each* angle of a regular polygon equals $\dfrac{(n-2)\,180°}{n}$. § 133.

6. Equal supplementary angles are *right* angles. § 28.

7. A straight line cannot intersect a circle in more than *one* point. § 185.

8. Two triangles with three *angles* of one equal to three *angles* of the other are congruent. § 80.

9. All angles of an *isosceles* triangle are always acute. §§ 69, 123.

10. If two chords intersect in a circle, the product of the segments of one is equal to the *square* of the segments of the other. § 333.

TEST SIX

Completing Statements

Complete the following statements.

1. The —— to the hypotenuse of a right triangle is equal to half the hypotenuse. § 159.

2. The angle made by a tangent and a secant drawn to a circle from an external point is measured by half the —— of the intercepted arcs. § 248.

3. The side opposite the 30° angle in a 30°–60° right triangle is equal to half the —— . § 160.

4. An —— angle of a triangle is equal to the sum of the two opposite interior angles. § 128.

5. The altitude drawn to the hypotenuse of a right triangle is a —— proportional between the segments of the hypotenuse. § 323.

6. If the diagonals of a quadrilateral bisect each other, the figure is a —— . § 148.

7. If one angle of a right triangle is 60°, the altitude on the hypotenuse is one half the side opposite the —— ° angle. § 160.

8. The locus of points equidistant from two intersecting lines and also equidistant from two other parallel lines may, at most, be —— points. §§ 257, 258.

9. The point of intersection of the perpendicular bisectors of the sides of a triangle is the center of the —— circle. § 271.

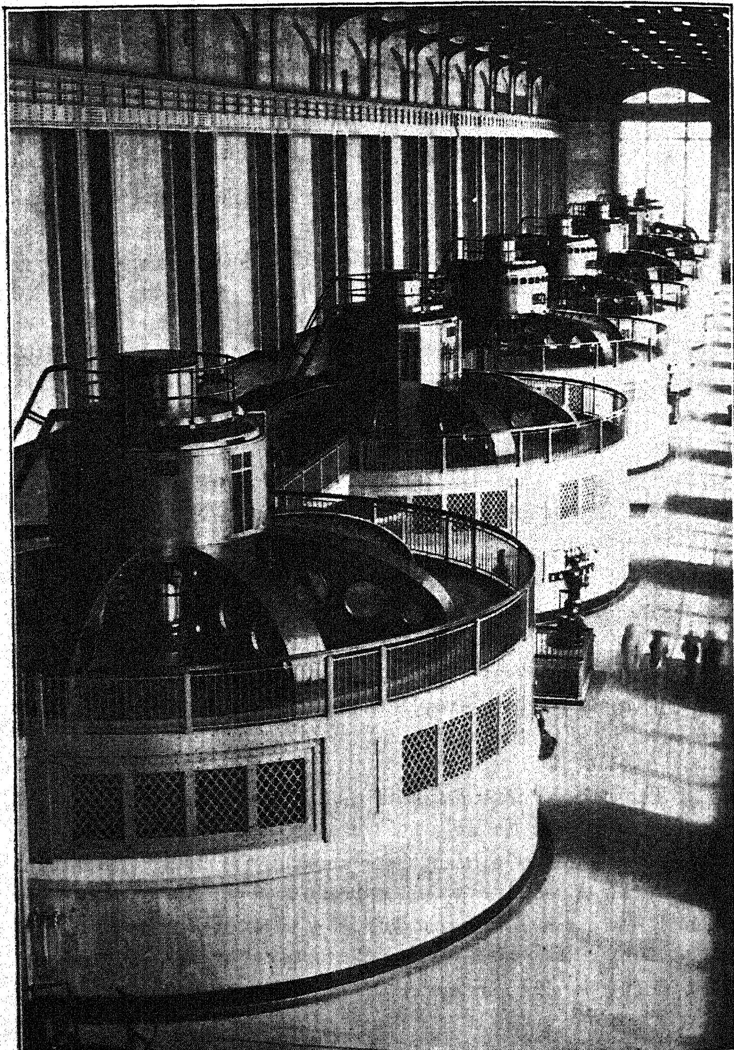
10. If two sides of a triangle are unequal, the angles opposite are unequal, and the smaller angle is opposite the —— side. § 170.

11. If two chords intersect within a circle the angle formed is measured by half the —— of the intercepted arcs. § 246.

12. If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are —— § 115.

13. If, from an external point, secants are drawn to a circle, the product of each secant by its external segment is a —— . § 336.

14. Two triangles are —— if their corresponding sides are proportional. § 314.



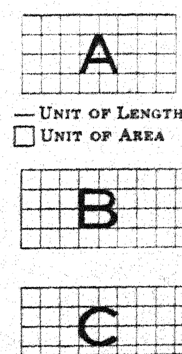
Geometry in Industry © Gendreau

These great machines and the building in which they are housed were designed on geometric principles.

UNIT SEVEN

AREAS OF POLYGONS

340. Area of a surface. In earlier units you have measured lengths, angles, and arcs. In each case you have used a unit of measure of the same kind as the quantity you were measuring. In the same way, in order to find the area of a surface, such as rectangle A, we must find how many times it contains a **unit of surface,** such as a unit square.



If A is 8 units long and 4 units wide, we can draw 4 rows with 8 squares in a row. Thus its area will be 4×8 or 32.

If B is 4 units wide and $8\frac{1}{2}$ units long, there will be 32 whole squares and 4 half squares. That is, $4 \times 8\frac{1}{2}$ or 34 in all.

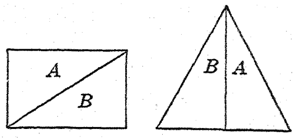
Even if the sides of the rectangle have a length of 8^+ and a width of 3^+, its area will be approximately that of the rectangle outlined within C or $8.^+ \times 3.^+ = 24.^+$

Hence we have:

341. Theorem. *The area of a rectangle is equal to the product of its base and altitude.*

HISTORICAL NOTE. — We can give, as Euclid, a full geometric proof of the theorem above, first showing that the areas of two rectangles having equal bases (altitudes) have the same ratio as their altitudes (bases). Then by comparing the rectangle with the unit square the theorem follows.

342. Polygons equal in area. While it is evident that *congruent* polygons are *equal in area*, polygons equal in area are not necessarily congruent.

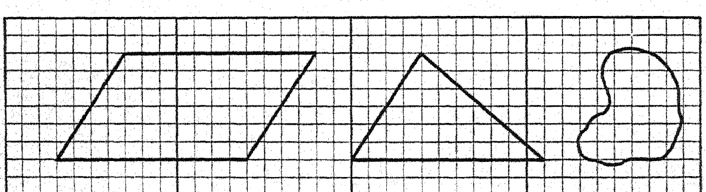


Thus, in the adjoining figures, the rectangle and the isosceles triangle have the same area because they are composed of the equal parts, A and B. But they are not congruent.

When there is no possibility of misunderstanding we may use the word *equal* to mean *equal in area.*

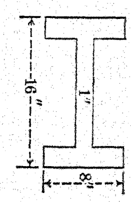
EXERCISES

1. Find the approximate areas of the figures below, using one of the small squares as the unit of area. In counting the squares, include a square in the figure if one half of it or more lies within the figure; otherwise do not include it.



2. The altitude of rectangle A is 10 ft. and of rectangle B is 12 ft. The base of each is 30 ft. Find the ratio of A to B.

3. Find the ratio of the areas of two rectangles whose dimensions are 8 ft. by 1 ft., and 10 ft. by 8 in., respectively.



4. All lots of a city block are 120 ft. deep. If a lot A in this block is 50 ft. wide and lot B is 60 ft. wide, compare the areas of the two lots (find their ratio in lowest terms).

5. Find the cross-sectional area of this I-beam if the width of the upper and lower rectangles is $1\frac{1}{2}$ in.

6. If one field is 80 rd. long and 40 rd. wide, and another field is 60 rd. square, find the ratio of their areas without computing the areas.

7. A rectangle is 50 in. wide and 200 in. long. Compare its perimeter with that of an equal square.

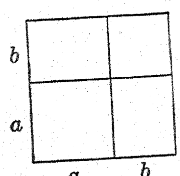
8. A fence incloses a field 25 rd. wide and 64 rd. long. How much greater area would the fence inclose if the field were square?

---

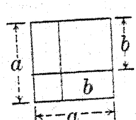
9. Show that two rectangles having equal bases are to each other as their altitudes.

HINT. — Assume b the common base, and a and a' the altitudes.

10. Two rectangles having equal altitudes are to each other as their bases.



I



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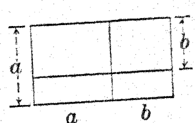
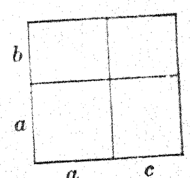


III



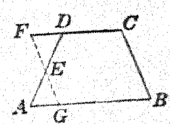
IV

11. Show geometrically that $(a+b)^2 = a^2 + 2ab + b^2$. (Fig. I.)

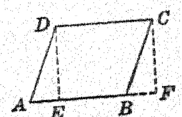
12. Show geometrically that $(a-b)^2 = a^2 - 2ab + b^2$. (Fig. II.)

13. Show geometrically that $(a+b)(a-b) = a^2 - b^2$. (Fig. III.)

14. Show geometrically that $(a+b)(a+c) = a^2 + ac + ab + bc$. (Fig. IV.)



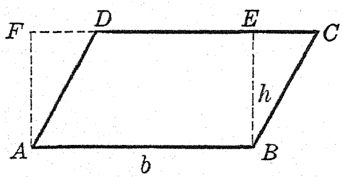
*15. In trapezoid $ABCD$, FG is drawn parallel to BC so that $EF = EG$. Prove that $ABCD = GBCF$.



*16. $ABCD$ is a parallelogram. DE and CF are drawn perpendicular to AB. Prove that $ABCD$ has the same area as $EFCD$.

PROPOSITION 1. THEOREM

343. *The area of a parallelogram is equal to the product of its base by its altitude.*



Given: $\square ABCD$ with base b and altitude h.

To prove: Area of $\square ABCD = hb$.

Plan: Show that $\triangle BCE \cong \triangle ADF$; hence $\square AC = \square AE$.

Proof:

STATEMENTS	REASONS
1. Draw $BE \perp CD$ and $AF \perp CD$ produced. $AF \parallel BE$.	1. § 107.
2. AE is a $\square$. Area $AE = hb$.	2. *Give reasons.*
3. $\triangle BCE \cong \triangle ADF$.	3. *Why?*
4. $ABED + \triangle BEC = ABED + \triangle ADF$.	4. *Ax.* 2.
5. $\square AC = \square AE$ and $\square AC = hb$.	5. *Ax.* 7.

344. COROLLARY 1. *Parallelograms having equal bases and equal altitudes are equal in area.*

345. COROLLARY 2. *Two parallelograms are to each other as the products of their bases and altitudes.*

346. COROLLARY 3. *Parallelograms having equal altitudes are to each other as their bases.*

347. COROLLARY 4. *Parallelograms having equal bases are to each other as their altitudes.*

EXERCISES

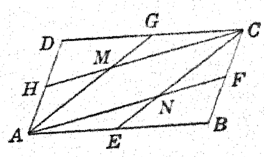
1. Find, using § 345, the ratio of the areas of two parallelograms with bases 10 in. and 16 in., and altitudes 8 in. and 12 in., respectively.

2. How many pieces of sod will it take to sod a lawn 42 ft. wide and 56 ft. long, if the pieces are 12 in. by 14 in.?

3. On a map drawn to the scale of 60 mi. to the inch, what area is inclosed in a strip 3 in. wide and 5 in. long?

4. Construct a parallelogram which shall be twice a given parallelogram.

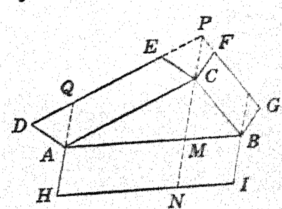
5. Construct a rectangle equal to half of a given parallelogram.



6. From two opposite vertices of parallelogram $ABCD$ line segments are drawn to the middle points of the sides. Prove that $AECG$ is a parallelogram and is half $\square ABCD$.

7. In Ex. 6, prove that $ANCM$ is a parallelogram.

8. Prove that $AN = \frac{2}{3} AF$ and $AM = \frac{2}{3} AG$. (See § 278.)



***9.** Using the results of Ex. 6–8 prove that $\square ANCM = \frac{1}{3} \square ABCD$.

***10.** Upon two sides of any triangle ABC the parallelograms $ACED$ and $BCFG$ are drawn. DE and GF are produced to intersect at P, and PC is drawn. Parallelogram $AHIB$ is drawn with side AH equal and parallel to PC. If HA produced cuts DE at Q, prove that $\square AE = \square AP$.

***11.** In Ex. 10, if PC produced cuts AB at M and HI at N, prove that $\square HM = \square AP$.

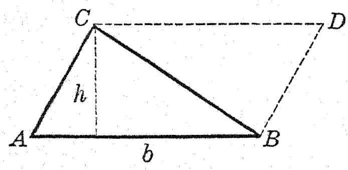
***12.** Using the results of Ex. 10 and 11 prove that $\square HB = \square AE + \square CG$.

***13.** Can you prove $\triangle ABC = \frac{1}{2} hb$?

HINT. — Form a parallelogram by drawing parallels to the opposite sides through C and B.

PROPOSITION 2. THEOREM

348. *The area of a triangle is equal to half the product of its base by its altitude.*



Given: Triangle ABC with base b, and altitude h.

To prove: $\triangle ABC = \frac{1}{2}hb$.

Plan: Form a ▱ and show that the △ is half the ▱.

Proof: Write in full.

349. COROLLARY 1. *Two triangles having equal bases and equal altitudes are equal in area.*

350. COROLLARY 2. *Two triangles are to each other as the products of their bases and altitudes.*

351. COROLLARY 3. *Triangles having equal altitudes are to each other as their bases.*

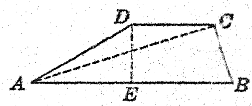
352. COROLLARY 4. *Triangles having equal bases are to each other as their altitudes.*

EXERCISES

1. Find the base of a triangle whose area is 288 sq. ft. and whose altitude is 9 ft.

2. Find the side of a square whose area equals that of a triangle with base 48 in. and altitude 24 in.

3. Find the ratio of the areas of triangles T and T' if they have equal bases and the altitudes are 18 in. and 15 in., respectively.



4. Trapezoid $ABCD$ has $AB = 24$ ft., $CD = 12$ ft., $\angle A = 30°$, and $AD = 10$ ft. Find its area by finding the area separately of triangles ABC and ACD.

5. What is the area of a rhombus whose diagonals are 24 in. and 30 in., respectively?

HINT. — Why are the diagonals perpendicular to each other? (§ 87.)

6. Prove that the area of a rhombus is equal to half the product of its diagonals.

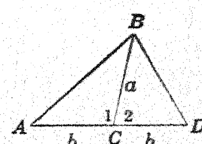
7. Prove that the diagonals of a parallelogram divide it into four equal triangles.

8. A triangle is half a parallelogram having the same base and altitude.

---

9. Find, to tenths, the altitude of an equilateral triangle whose side is 6 in. (See § 326.)

*10. Prove that the area of an equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$. Find the area if $s = 20$ in.



11. Prove that two triangles are equal if two sides of one are equal to two sides of the other, and the included angles are supplementary.

SUGGESTION. — Place the triangles with the supplementary angles adjacent, as in the figure.

12. Prove that the line segments joining the middle points of the sides of any triangle divide the triangle into four equal triangles.

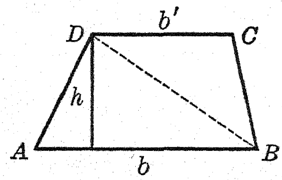
13. If, from the middle point of either diagonal of any quadrilateral, segments are drawn to the two opposite vertices, prove that they divide the quadrilateral into two equal quadrilaterals.

14. If, from the middle point of any side of a triangle, straight lines are drawn parallel to the other two sides, prove that the parallelogram thus formed is equal to half the triangle.

*15. Prove that the area of a triangle equals half the product of its perimeter and the radius of the inscribed circle.

Proposition 3. Theorem

353. *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*



Given: Trapezoid $ABCD$ with bases b and b', and altitude h.

To prove: Trapezoid $ABCD = \frac{1}{2}h(b + b')$.

Plan: 1. If b' is the base of $\triangle BCD$, what is its altitude? 2. Add the areas of $\triangle$s ABD and BCD.

Proof:

STATEMENTS	REASONS
1. Draw BD. The area of $\triangle ABD = \frac{1}{2}bh$, and the area of $\triangle BCD = \frac{1}{2}b'h$.	1. § 348.
2. $\triangle ABD + \triangle BCD = \frac{1}{2}bh + \frac{1}{2}b'h = \frac{1}{2}h(b + b')$.	2. *Ax.* 2.
3. Trapezoid $ABCD = \frac{1}{2}h(b + b')$.	3. *Ax.* 7.

354. B. **Theorem.** *The area of a trapezoid is equal to the product of its altitude and the segment connecting the mid-points of the legs.* (See § 157.)

EXERCISES

1. Find the area of a trapezoid whose bases are 17 in. and 23 in. and whose altitude is 15 in.

2. Find the area of a trapezoid the sum of whose bases is 45 in. and whose altitude is 10 in.

3. The area of a trapezoid is 1701 sq. yd., the altitude is 42 yd., and one base 36 yd. Find the other base.

4. A canal is 28 ft. deep, 120 ft. wide at the top, and 90 ft. wide at the bottom. What is the area of a cross section of it?

5. One base of a trapezoid is 10 ft., the altitude 4 ft., and the area 32 sq. ft. Find the length of the segment drawn between the non-parallel sides, parallel to the given base and 1 ft. from it.

HINT. — See §§ 156, 157.

6. Any trapezoid is bisected by the line segment joining the middle points of its bases.

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7. A parallelogram is bisected by any straight line drawn through the intersection of the diagonals.

8. The triangle having one of the non-parallel sides of a trapezoid as base and the middle point of the opposite side as vertex is equal to one half of the trapezoid.

9. The area of a trapezoid is equal to the product of one of its non-parallel sides and the distance to it from the middle point of the opposite side. (See Ex. 8.)

10. Through a given point draw a straight line that shall bisect a given parallelogram. (See Ex. 7.)

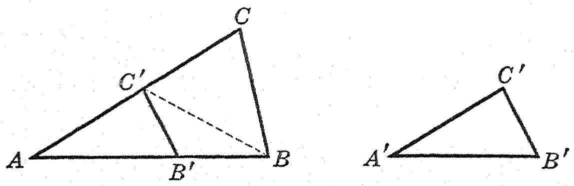
11. Draw a straight line parallel to a given straight line and which shall bisect a given parallelogram. (See Ex. 7.)

*12. Find the area of a trapezoid $ABCD$, if $AB = 30$ in., $AD = 10$ in., $\angle A = 30°$, and $\angle B = 45°$. (See § 160.)

355. A. Continuity. In §§ 341, 343, 348, and 353 you have had theorems about the areas of certain geometric figures. See if you can show how each of them can be made to depend on the statement: *If two sides (at least) of a quadrilateral are parallel, the area is equal to half the sum of the parallel sides multiplied by the perpendicular distance between them.*

PROPOSITION 4. THEOREM

356. B. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Given: Triangles ABC and $A'B'C'$ with $\angle A = \angle A'$.

To prove: $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times AC}{A'B' \times A'C'}$.

Plan: Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$. Compare the areas of △ $AB'C'$ and ABC'; also of △ ABC' and ABC. Recall § 351 and multiply.

Proof: For you to write.

EXERCISES

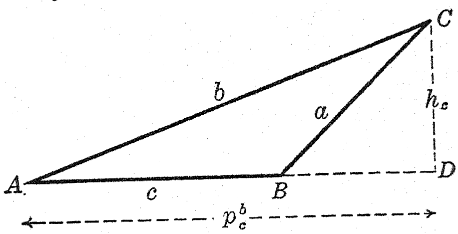
1. Two triangles that have an angle of one equal to an angle of the other, have the sides including the equal angles 4 in. and 9 in., and 12 in. and 5 in., respectively. Compare their areas.

2. In the figure of § 356, $AC = 12$ in., $AB = 9$ in., $A'C' = 18$ in. If the area of $\triangle ABC$ is 256 sq. in. and of $\triangle A'B'C'$ is 240 sq. in., find $A'B'$.

***3.** Two corresponding sides of similar triangles are 8 in. and 12 in., respectively. If the area of the smaller triangle is 48 sq. in., find the area of the larger.

***4.** Two triangles having an angle of one supplementary to an angle of the other are to each other as the products of the sides including those angles.

357. A. *A formula for the altitude of a triangle.*



Given: Triangle with sides a, b, and c, altitude h_c on c, s the semi-perimeter, $\angle A$ an acute angle.

To prove: $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$

Proof:

STATEMENTS	REASONS
1. $h_c^2 = b^2 - AD^2$.	1. *Why?*
2. $AD = p_c^b = \dfrac{b^2 + c^2 - a^2}{2c}$.	2. § 331.
3. $h_c^2 = b^2 - \left(\dfrac{b^2+c^2-a^2}{2c}\right)^2$.	3. *Why?*

Reducing to a fraction and factoring

4. $h_c^2 = \dfrac{4b^2c^2 - (b^2+c^2-a^2)^2}{4c^2}$

$= \dfrac{(a+b+c)(a+b-c)(b+c-a)(a-b+c)}{4c^2}$

5. Since $2s = a + b + c$, then $2(s-a) = b + c - a$, $2(s-b) = a - b + c$, and $2(s-c) = a + b - c$.

Substituting in (4)

6. $h_c^2 = \dfrac{4s(s-a)(s-b)(s-c)}{c^2}$ or,

$h_c = \dfrac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$.

358. Hero's formula. Since the area of a triangle is $A = \frac{1}{2}bh$, from § 357 we have,

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Historical Note. — This formula was first given by Hero, an Egyptian mathematician who lived in Alexandria about the beginning of the Christian era.

EXERCISES

In Ex. 1–8, find s, $s-a$, $s-b$, and $s-c$.

1. $a = 5$ in., $b = 13$ in., $c = 12$ in.

2. $a = 7$ in., $b = 24$ in., $c = 25$ in.

3. $a = 17$ in., $b = 15$ in., $c = 8$ in.

4. $a = 60$ ft., $b = 11$ ft., $c = 61$ ft.

5. $a = 21$ in., $b = 29$ in., $c = 20$ in.

6. $a = 13$ ft., $b = 15$ ft., $c = 14$ ft.

7. $a = 20$ ft., $b = 13$ ft., $c = 21$ ft.

8. $a = 25$ ft., $b = 36$ ft., $c = 29$ ft.

9.–16. Find the area of each of the triangles in Ex. 1–8.

17. Find the altitude on side c in the triangle with side $a = 25$ in., $b = 17$ in., $c = 28$ in.

18. Find the altitude on side c in the triangle with sides $a = 39$ ft., $b = 17$ ft., $c = 44$ ft.

19. Find the altitude on side a in the triangle with sides $a = 17$ in.. $b = 26$ in., $c = 25$ in.

20. Find the altitude on side b in the triangle with sides $a = 41$ ft. $b = 51$ ft., $c = 58$ ft.

---

21. Find correct to one decimal place the altitude on side c if $a =$ 21 in., $b = 20$ in., and $c = 29$ in.

22. Find to one decimal place the area of a triangle whose sides are 20 ft., 24 ft., and 28 ft.

23. If a side of an equilateral triangle is 8 ft., find the area.

24. If a is a side of an equilateral triangle, find the formula for the area by the formula in § 358.

*25. Find the area of a parallelogram whose sides are 6 in. and 8 in. and whose diagonal is 12 in.

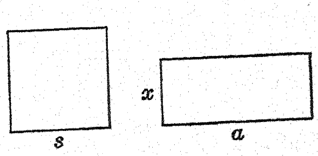
*26. Explain how the area of a trapezoid can be found, if you are given the bases and one leg, and if the given leg makes an angle of 30° with the base.

*27. Find the area of a trapezoid if its non-parallel sides are 15 ft. and 18 ft. and its bases are 20 ft. and 32 ft.

HINT. — Find the sides of the similar triangles formed by producing the legs. Then find their areas by § 358 and subtract.

359. Algebra in geometric constructions.

1. Construct a rectangle equal to a given square and having its side equal to a given segment.



SOLUTION. — If the side of the given square is s, and the given segment is a, let the other side of the rectangle be x.

Then the area of the square is s^2, and the area of the rectangle is ax. The problem requires that:

$$s^2 = ax$$

Hence, by § 289–6, $a : s = s : x$

Therefore construct x by § 297.

2. Construct a square equal to a given parallelogram.

HINT. — $x^2 = hb$. Hence $h : x = x : b$. Construct x the mean proportional between h and b. (§ 337.)

3. Construct a square having half the area of a given square.

HINT. — $x^2 = \frac{1}{2} s^2$. Hence $s : x = x : \frac{1}{2} s$.

4. Construct a triangle equal to a given triangle, and having one side equal to a given segment.

HINT. — Take the given segment as the base.

5. Construct a right triangle equal to a given triangle, and having one of its legs equal to a given segment.

6. Construct a rectangle having a given side and equal to a given rectangle.

7. Construct a rectangle having a given side and equal to a given parallelogram.

8. Construct an isosceles triangle equal to a given triangle and on the same base.

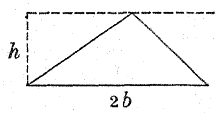
9. Construct a rhombus having a given diagonal and equal to a given parallelogram. (See Ex. 6, § 352.)

10. Bisect a given triangle by a line drawn through any vertex.

11. Divide a given triangle into any ratio by a straight line drawn through any vertex.

12. Construct a triangle having two of its sides equal to segments a and b and equal to a given triangle. (Take b as the base.)

13. Construct a triangle having a given angle and equal to a given parallelogram.



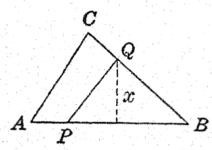
HINT. — If the base of the ▱ is b and its altitude is h, what is its area? Take $2b$ for the base of the △.

14. Construct a right triangle equal to a given right triangle and having its hypotenuse equal to a given segment. (Recall Locus VI, § 260.)

15. Construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

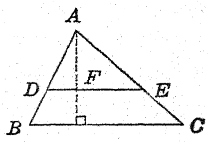
*16. Construct an equilateral triangle equal to a given triangle. (See Ex. 10, § 352.)

*17. Bisect a given triangle by drawing a straight line through a given point on one side of the triangle.



HINT. — Let $\triangle ABC$ be the given triangle with altitude h and base b, and P the given point. If PQ be assumed to bisect the triangle and if x is $\perp PB$, then by the conditions of the problem $\triangle PQB = \frac{1}{2}\triangle ABC$, or $PB \times x = \frac{1}{2}\ hb$. Hence $PB : \frac{1}{2}h = b : x$.

*18. Bisect a given triangle by drawing a straight line parallel to one of the sides.



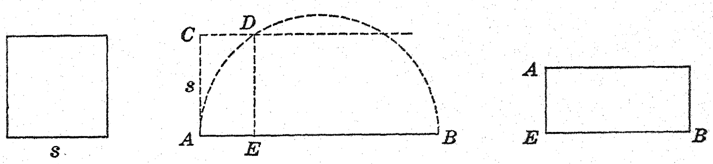
HINT. — Let the area of $\triangle ABC$ be $\frac{1}{2}hb$. Then $\frac{1}{2}hb = AF \cdot DE$. Let $AF = x$ and $DE = y$:

Then (1) $xy = \frac{1}{2}hb$, and, since $\triangle ABC \sim \triangle ADE$, $x : y = h : b$ or

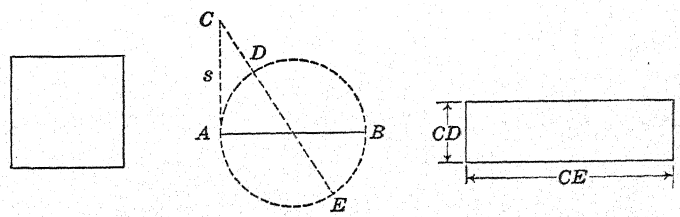
(2) $x = \dfrac{hy}{b}$. Substitute this value in (1) and simplify and we have $y^2 = \frac{1}{2}b^2$. Hence $b : y = y : \frac{1}{2}b$.

*19. Construct a rectangle equal to a given square, and having the sum of its base and altitude equal to a given segment.

HINT. — If AB is the given segment and s a side of the given square, make $AC = s$.

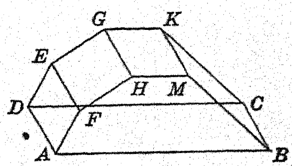


*20. Construct a rectangle equal to a given square, and having the difference of its base and altitude equal to a given segment.

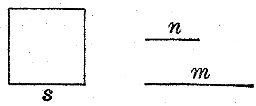


*21. If $ABCD$ is any trapezoid, and the diagonals AC and BD intersect at O, then $\triangle AOD = \triangle BOC$.

*22. Prove that $\square AFED + \square FHGE + \square HMKG + \square MBCK = \square ABCD$.



*21. Construct a square having a given ratio to a given square.



HINT. — Let it be required to construct a square with side x so that $x^2 : s^2 = m : n$.

Then $\quad x = \sqrt{\frac{s^2}{n} \times m.}$

Let $\quad y = \frac{s^2}{n}$, then $x = \sqrt{ym.}$

Thus, x is the mean proportional between y and m.

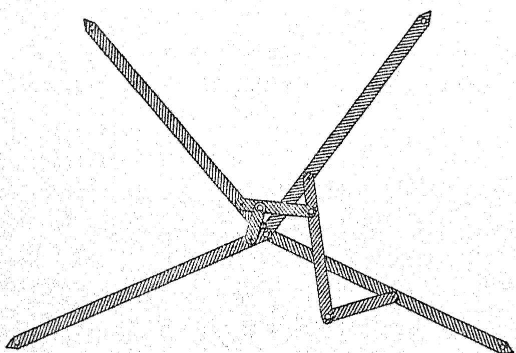
To construct y we have

$$y = \frac{s^2}{n}, \text{ or } ny = s^2.$$

Hence, $n : s = s : y$.

Therefore, first construct y the third proportional to n and s. Then construct x the mean proportional between y and m.

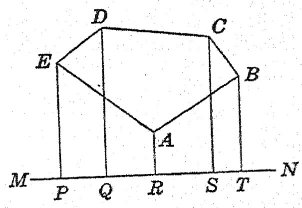
HISTORICAL NOTE. — History tells us of the effort to solve three famous problems called "The three famous problems of antiquity." They were: (1) Trisecting any angle. (2) Squaring the circle. (3) Duplicating the cube.



Mathematicians long endeavored to solve them with the instruments of elementary geometry, which are the unmarked straight-edge and compasses. In the 19th century their solutions by these instruments was proved impossible. All three problems can, however, be solved in several ways by using other instruments. Thus the *linkage* shown will trisect any angle.

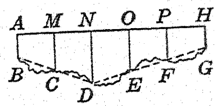
PRACTICAL APPLICATIONS

(OPTIONAL)



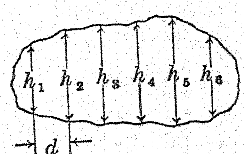
1. Surveyors sometimes find the area of a tract of land as follows: A base line MN is staked off, and from the various points in the boundary, the distances to this line are then measured, as EP, DQ, etc., and the distances PQ, QR, etc., are also measured. Show how to compute the area of $ABCDE$.

If $EP = 2000$ ft., $DQ = 2500$ ft., $CS = 2300$ ft., $BT = 1800$ ft., $AR = 800$ ft., $PQ = 800$ ft., $QS = 1900$ ft., $ST = 500$ ft., $TR = 1700$ ft., compute the area.

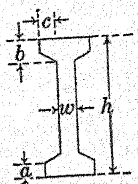


2. In order to determine the flow of water in a stream, the area of a cross section $ABCDEFGH$ of the stream, at right angles to the current, is first found as follows: Soundings are taken at A, M, N, etc., and the areas of trapezoids $ABCM$, $MCDN$, etc., are computed and added.

If $AM = MN = NO = OP = PH = 10$ ft., $AB = 5$ ft., $MC = 9$ ft., $ND = 12$ ft., $OE = 9$ ft., $PF = 7$ ft., and $HG = 5$ ft., find the area of the cross section. This is known as the *trapezoidal rule* for finding an area. It is used in measuring the area of land bounded on one side by an irregular line.



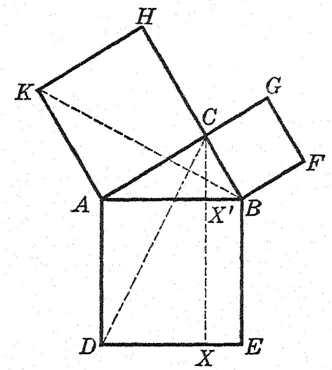
3. A fairly accurate method used for computing the area bounded by an irregular curve, known as the *mean ordinate* method, is as follows: At equal distances d measure the widths h_1, h_2, h_3, etc., of the area inclosed. Show that if a large number of widths are measured, a close approximation to the area is given by $d(h_1 + h_2 + h_3 + \cdots)$.



4. The formula Area $= wh + 2\,c(a + b)$ for computing the area of the cross section of an I-beam, where w, h, c, a, and b are the dimensions shown in the drawing, is given in books on engineering. Prove this formula.

PROPOSITION 5. THEOREM

360. B. *In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.*



Given: Right $\triangle ABC$, with $\angle C$ a right angle. BG, AE, and AH are squares.

To prove: Area AE = area BG + area AH.

Plan: Show that rectangle AX and square AH are equal in area to twice the congruent $\triangle ACD$ and ABK, respectively.

Proof:

STATEMENTS	REASONS
1. Draw $CX \perp DE$, intersecting AB in X'. Draw KB and CD. HCB is a straight line.	1. § 27.
2. AK is the common base and AC is equal to the altitude of square AH and of $\triangle ABK$.	2. § 145.
3. Area $\triangle ABK = \frac{1}{2}$ area AH.	3. *Why?*

STATEMENTS	REASONS
4. AD is the common base and AX' is equal to the altitude of $\square AX$ and $\triangle ACD$.	4. *Why?*
5. Area $\triangle ACD = \frac{1}{2}$ area AX.	5. *Why?*
6. $\triangle ABK \cong \triangle ACD$.	6. *Prove.*
7. $\square AX$ = square AH.	7. *Ax.* 1.

Complete, drawing CE and AF; show that $BX = BG$.

EXERCISES

(OPTIONAL)

Prove the Pythagorean theorem by using the following figures:

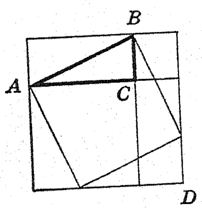


FIG. 1

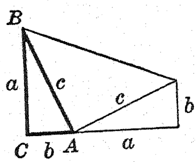


FIG. 2

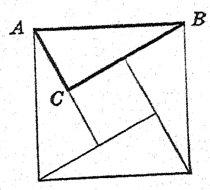


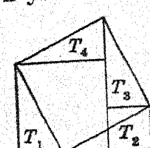
FIG. 3

1. In Fig. 1, if the equal rectangles AB and CD are taken away, what remains? If the four congruent triangles at the corners are taken away, what remains? Give the proof in full.

2. Fig. 2 suggests a proof given by James A. Garfield, once President of the United States. Give his proof in full.

SUGGESTION. — Compare the area of the trapezoid with the sum of the areas of the triangles.

3. Fig. 3 is a drawing used by Bhaskara, who was born in India 1114 A.D. We do not know what his proof was. See if you can find one to fit the figure.

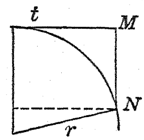


4. In the figure prove the theorem by showing that triangles T_1, T_2, T_3, and T_4 are congruent. If T_1 and T_2 are subtracted from the figure, what remains? What remains if T_3 and T_4 are subtracted?

5. Prove the Pythagorean theorem by using the results of Ex. 12, § 347 when angle C is a right angle.

NOTE. — The extension of the Pythagorean theorem given in Ex. 10–12, § 347, is due to Pappus, a Greek geometer, who lived about 300 A.D.

6. Railroad curves are sometimes laid out by measuring offsets from the tangent line at the beginning of the curve. Show that if r is the radius of the curve and t the distance along the tangent from the beginning of the curve to any offset MN, then the offset $MN = r - \sqrt{r^2 - t^2}$.



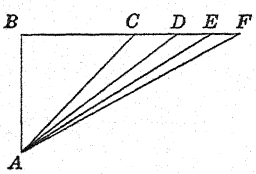


FIG. 1

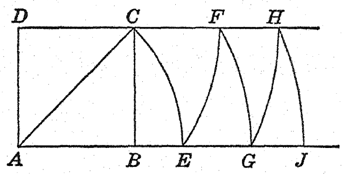
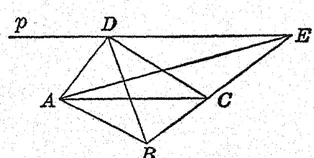


FIG. 2

7. In Fig. 1, if $AB = BC = 1$, $BD = AC$, $BE = AD$, and $BF = AE$, find the length of AC, AD, AE, and AF.

8. In Fig. 2, if AC is a square with $AB = 1$, and $AC = AE$, $DE = DF$, $AF = AG$, $DG = DH$, and $AH = AJ$, find AE, DF, AG, DH, and AJ.

361. To transform a polygon into a triangle means to construct a triangle having the same area as the polygon.

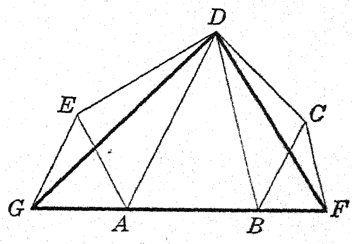


If $p \parallel AC$, will any triangle with base AC and vertex on p have the same area as $\triangle ACD$?

If BC produced intersects p at E, will $\triangle ABE$ have the same area as $ABCD$? Then how would you transform a quadrilateral into a triangle? Can you think of a way to transform a pentagon into a quadrilateral? Into a triangle?

CONSTRUCTION XXI

362. *To construct a triangle equal in area to a given polygon.*



Given: Polygon $ABCDE$.

Required: Construct a $\triangle$ equal in area to $ABCDE$.

Plan: Draw $CF \parallel BD$ and replace $\triangle BCD$ by $\triangle BFD$. Proceed in a similar way with $\triangle AED$.

Construction: 1. Draw diagonal BD, and draw $CF \parallel BD$ meeting AB produced at F. Draw DF.

2. Draw diagonal AD, and draw $EG \parallel AD$ meeting BA produced in G. Draw DG.

3. $\triangle GFD$ is the required $\triangle$.

Proof: Prove $\triangle BCD = \triangle BFD$ and $\triangle AED = \triangle AGD$ (§ 349).

Write the proof in full.

EXERCISES

1. Using the method of § 362, construct a triangle equal to a given rectangle.

2. Construct a square equal to a given triangle.

HINT. — Let the triangle $= \frac{1}{2}hb$. Then $x^2 = \frac{1}{2}hb$, or $\frac{1}{2}h : x = x : b$.

3. Construct a square equal to a given polygon.

HINT. — Construct a triangle equal to the polygon, then use Ex. 2.

4. Construct a square equal to the sum of two given squares.

SUGGESTION. — If a side of one of the given squares is a, and of the other b, construct a right triangle with legs a and b. Why is the hypotenuse c the side of the required square?

5. Construct a square equal to the difference of two given squares.

6. Construct a triangle equal to an irregular quadrilateral, and, by measuring the base and altitude of the triangle, compute the area of the quadrilateral.

---

7. Construct a square equal to the sum of three given squares.

8. How would you construct a square equal to the sum of any number of given squares?

9. Construct a square equal to twice a given square.

10. Circumscribe about a given circle a triangle similar to a given triangle.

HINT. — Inscribe a circle O in the given triangle ABC and use the central angles.

11. Construct $x = \sqrt{a^2 + b^2}$.

12. Construct $x = \sqrt{a^2 - b^2}$.

*13. Construct a triangle equal to the sum of two given triangles.

HINT. — See Ex. 2 and Ex. 4.

*14. Construct a triangle similar to a given triangle and having a given altitude.

363. B. Some important relations are collected in the following exercises. A few of them you have had before. Prove each one.

EXERCISES

1. The altitude of an equilateral triangle with side a is $\frac{a}{2}\sqrt{3}$.

2. If d and d' are the diagonals of a rhombus, its area is $\frac{1}{2}\,dd'$.

3. The area of a right triangle whose legs are a and b is $\frac{1}{2}\,ab$.

4. The diagonal of a square with side a is $a\sqrt{2}$.

5. The area of an equilateral triangle with side a is $\frac{a^2}{4}\sqrt{3}$.

6. The radius of the circle inscribed in an equilateral triangle is one-third of the altitude.

7. The radius of the circle circumscribed about an equilateral triangle is two-thirds the altitude.

8. The altitudes of an equilateral triangle bisect the angles and the opposite sides of the triangle. Hence the altitude from any vertex is equal to the median, to the angle bisector drawn from that vertex, and to the perpendicular bisector of the opposite side. Therefore the incenter, circumcenter, orthocenter, and center of gravity coincide in a point of trisection of the altitude.

9. The area of a triangle circumscribed about a circle is equal to the perimeter of the triangle times half the radius of the circle.

10. The area of any polygon circumscribed about a circle is equal to the perimeter of the polygon times half the radius of the circle.

11. If r is the radius of a circle inscribed in an equilateral triangle, the area of the triangle is $3\,r^2\sqrt{3}$.

12. Find a formula for the area of an equilateral triangle inscribed in a circle in terms of the radius R of the circle.

MISCELLANEOUS EXERCISES

In Ex. 1–5 use the formulas in Ex. 1–5 above.

1. Find the area of a rhombus with diagonals 18 ft. and 20 ft. (Ex. 2.)

2. Find the area of a right triangle with legs 12 ft. and 19 ft. (Ex. 3.)

3. Find the area of an equilateral triangle with side 12 in. (Ex. 5.)

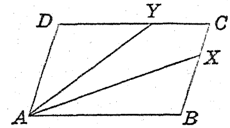
4. What is the altitude of the triangle in Ex. 3? (Ex. 1.)

5. The side of a square is 16 in. Find the diagonal correct to the nearest tenth. (Ex. 4.)

6. A rope attached to a flag pole is 8 ft. longer than the pole. When stretched out it just reaches the ground, 20 ft. from the pole. Find the height of the flag pole, assuming that the ground at its foot is level.

---

7. The oldest mathematical book extant, a papyrus copied by the Egyptian scribe Ahmes about 1700 B.C. from a still older book, gives for an isosceles triangle whose equal sides are 10 *ruths* and base 4 *ruths*, the area of 20 square *ruths*. By what rule must this area have been computed? What is its true area?



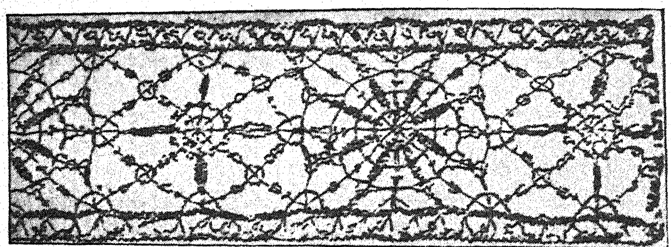
*8. Trisect any given parallelogram by straight lines drawn through any vertex.

HINT. — If $BX = \frac{2}{3} BC$ and $DY = \frac{2}{3} DC$, show that $\triangle ABX = \triangle ADY = \frac{1}{3} ABCD$.

*9. Construct a parallelogram having a given altitude and equal to a given trapezoid.

*10. Construct a square equal to a given trapezoid.

*11. If from the point of intersection of the medians of any triangle segments are drawn to the three vertices, prove that they form with the sides three equal triangles.



REGULAR POLYGONS IN LACE PATTERNS

364. A series of equal ratios. If a certain polygon P has sides of 2, 8, 4, and 6 and a similar polygon P' has sides 3, 12, 6, and 9 corresponding, respectively, to the sides of P, what is the ratio of similitude?

What is the ratio of the perimeters of the two polygons? Is it equal to the ratio of similitude?

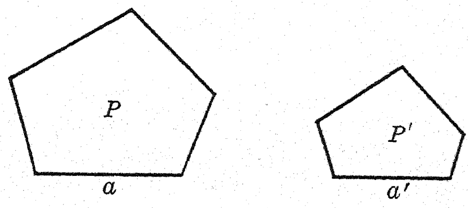
If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, do you think $\frac{a+c+e}{b+d+f} = \frac{a}{b}$? Clearing of fractions, why is $ab + bc + be = ab + ad + af$? (Compare each term on the left with the corresponding one on the right. Why is $bc = ad$? $be = af$?)

See if you can prove:

In a series of equal ratios, the sum of the numerators is to the sum of the denominators as any numerator is to its denominator.

PROPOSITION 6. THEOREM

365. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Given: Similar polygons P and P', with perimeters p and p', and corresponding sides a, b, c, etc., and a', b', c', etc.

To prove: $\frac{p}{p'} = \frac{a}{a'}$.

Analysis: 1. What series of equal ratios is there in similar polygons P and P'?

2. Express the proportions you are required to prove in terms of the sides of the polygon and use § 364.

Proof: Left for you to write out.

EXERCISES

1. The perimeters of two similar polygons are 144 yd. and 256 yd., respectively. A side of the first is 18 yd. Find the corresponding side of the second.

2. The sides of a polygon are 8 in., 12 in., 15 in., 6 in., and 20 in. Find the perimeter of a similar polygon whose longest side is 15 in.

3. The perimeter of a rectangle is 44 in. and the ratio of two of the sides is $\frac{5}{6}$. Find the perimeter of a similar rectangle if its larger side is 15 in.

4. The perimeter of an isosceles triangle is 66 in., and the ratio of a leg to the base is $\frac{3}{5}$. Find the perimeter of a similar triangle whose base is 25 in.

---

5. Is the converse of the theorem in § 365 true?

6. Prove that the perimeters of two similar triangles have the same ratio as any two corresponding medians, altitudes, or angle bisectors.

7. The perimeter of a polygon is p and one side is x. If the perimeter of a similar polygon is p', find the side corresponding to x.

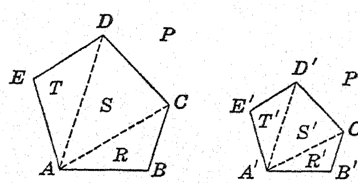
8. A rectangular field is w yd. wide and l yd. long. Find the perimeter of a similar field $3w$ yd. wide.

9. Prove that if two polygons are each similar to a third polygon, they are similar to each other.

*10. If two quadrilaterals are similar, do the diagonals from corresponding vertices divide the quadrilaterals into pairs of similar triangles? Prove it.

*11. Prove that two corresponding diagonals of two similar polygons have the same ratio as a pair of corresponding sides.

*12. Does a line parallel to the bases of a trapezoid divide it into two similar trapezoids? Prove that your answer is correct.



366. Similar polygons can be divided into similar triangles. If polygon $P \sim$ polygon P', and diagonals are drawn in each from corresponding vertices A and A', see if you can prove $\triangle R \sim \triangle R'$, $\triangle S \sim \triangle S'$, $\triangle T \sim \triangle T'$.

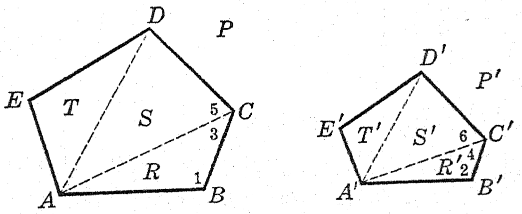
SUGGESTIONS. — What methods are available for proving $\triangle R \sim \triangle R'$? (See § 339.)

What does the fact that polygon $P \sim$ polygon P' tell you about the corresponding sides and angles of P and P'?

If you cannot prove the corresponding triangles similar see the next section.

Proposition 7. Theorem

367. *If two polygons are similar they can be divided into triangles which are similar and similarly placed.*



Given: Polygon $P \sim$ polygon P', A and A' being corresponding vertices.

To prove: P and P' can be divided into similar $\triangle$, similarly placed.

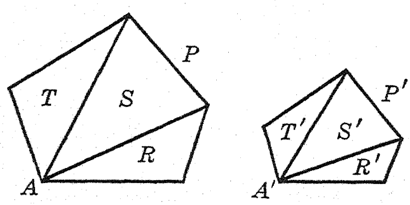
Plan: Show that $\triangle R \sim \triangle R'$, $\triangle S \sim \triangle S'$, $\triangle T \sim \triangle T'$ and that they are in the same order.

Proof:

STATEMENTS	REASONS
1. Draw diagonals from A and A' forming $\triangle$ R, S, and T, counterclockwise about A; and R', S', T' in the same order about A'. In $\triangle$ R and R', $\angle 1 = \angle 2$ and $AB : A'B' = BC : B'C'$.	1. *Given and* § 302.
2. $\triangle R \sim \triangle R'$.	2. § 312.
3. $\angle C = \angle C'$, and $\angle 3 = \angle 4$.	3. § 302.
4. $\angle 5 = \angle 6$.	4. *Ax.* 3.
5. $BC : B'C' = AC : A'C' = CD : C'D'$.	5. § 302.
6. $\therefore \triangle S \sim \triangle S'$.	6. § 312.
7. $\triangle T \sim \triangle T'$.	7. *By a similar proof.*

PROPOSITION 8. THEOREM

368. *If two polygons can be divided into triangles which are similar and similarly placed, the polygons are similar.*



Given: Polygons P and P' divided by diagonals from corresponding vertices A and A' into ⧍; $R \sim R'$, $S \sim S'$, $T \sim T'$; the ⧍ taken counterclockwise about A and A'.

To prove: $P \sim P'$.

Analysis: Think: "To show that $P \sim P'$ I must show that corresponding ∠s are equal and that corresponding sides are proportional.

Since the ⧍ are given similar, I know ——"

Proof: Write out.

EXERCISES

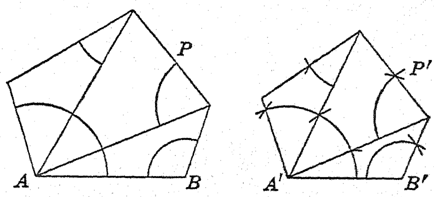
1. Given quadrilateral $ABCD$ with diagonal AC. On opposite sides of a given segment $A'C'$ corresponding to AC, construct triangle $A'C'D' \sim \triangle ACD$, and $\triangle A'C'B' \sim \triangle ACB$. Is $A'B'C'D' \sim ABCD$? Prove it.

2. Construct a pentagon $A'B'C'D'E'$ similar to a given pentagon $ABCDE$, given segment $A'B'$ corresponding to segment AB.

HINT. — Using the figure of § 367 construct $\triangle R' \sim \triangle R$, $\triangle S' \sim \triangle S$, and $\triangle T' \sim \triangle T$ by constructing equal angles.

Construction XXII

369. *To construct on a given segment as side, a polygon similar to a given polygon.*



Given: Polygon P and segment $A'B'$, corresponding to side AB of P.

Required: On $A'B'$ construct a polygon similar to P.

Write out the construction and proof.

EXERCISES

1. In finding the distance AP of an inaccessible point P a base line AB was taken and the following measurements were made: $AB =$ 225 yd., $\angle BAP = 105°$, $\angle ABP = 40°$. Make a drawing with a scale of 1 in. = 100 yd. and compute the distance AP.

2. Given any $\triangle ABC$, by copying its angles construct a $\triangle A'B'C'$ similar to $\triangle ABC$, and having one of its sides equal to a given segment.

3. Find the area of an isosceles triangle whose vertex angle is 30° and whose legs are each 18 in.

Hint. — Draw an altitude to the leg and use § 160.

4. Find the area of $\triangle ABC$ if $\angle A = 150°$, side $AC = 12$ in., and side $AB = 18$ in.

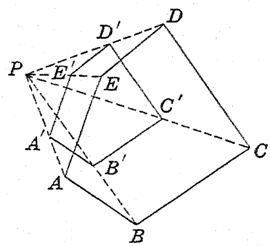
Hint. — Draw an altitude from C to BA extended. Find the altitude by § 160.

---

5. Find the ratio of the areas of two isosceles triangle each having a vertex angle of 62°, if the legs of one are each 12 ft. and the legs of the other 16 ft.

Hint. — See § 356.

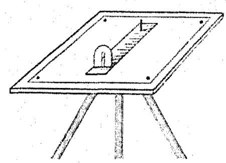
6. Find the area of an isosceles triangle whose vertex angle is 135° if the legs are each 20 in.



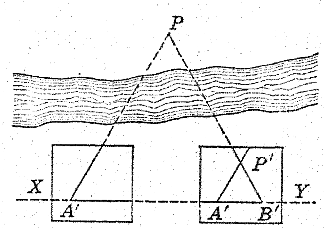
*7. The vertices of a given polygon $ABCDE$ are joined to a given point P. A segment $A'B'$ is drawn parallel to AB so that A' is on PA and B' on PB. Then $B'C' \parallel BC$, $C'D' \parallel CD$, $D'E' \parallel DE$, and $E'A'$ is drawn as in the figure. Prove $A'B'C'D'E' \sim ABCDE$.

Suggestion. — How may $E'A'$ be proved parallel to EA? Derive a proportion to show that $PE : PE' = PA : PA'$.

370. The plane table. A plane table consists of a drawing board. A sheet of paper is fastened to the board with thumb tacks, and a straight edge is laid on it for sighting and drawing lines. The instrument is used for finding the distances between inaccessible points and for making maps of small areas where only rough approximation is required.



To find the distance AP of the inaccessible point P, a triangle similar to $\triangle ABP$ is constructed as follows:



1. Lay out a base line XY and setting the plane table at point A, sight the straight edge toward Y and draw a line along it from A.

2. Then sight and draw a line toward P.

3. Move the plane table to B and, after sighting along your base line toward X, sight and draw a line toward P.

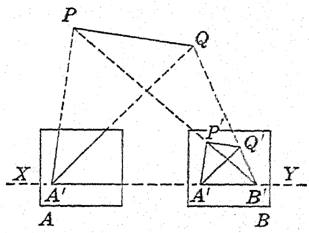
4. The triangle on your plane table is similar to $\triangle ABP$. The ratio of similitude may be found by measuring $A'B'$ very accurately and comparing it with AB.

EXERCISES

(OPTIONAL)

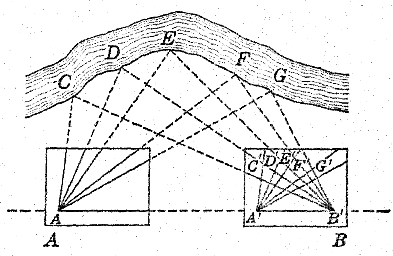
1. In the figure of § 370, if $AB = 350$ ft., $A'B' = 20$ in., and $A'P' = 23\frac{3}{8}$ in., find AP.

2. Study the figure and explain in their order the steps in finding the inaccessible distance PQ by use of the plane table.



3. In Ex. 2 prove that PQ may be found by means of the proportion $PQ : P'Q' = AB : A'B'$.

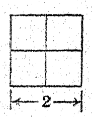
4. The following drawing shows the method of making a map $C'D'E'F'G'$ of the river bank $CDEFG$ by use of the plane table. Study the drawing, and explain how the points C', D', E', F', and G' of the map are obtained as the intersections of corresponding lines drawn to the points C, D, E, F, G from two points in the base line.



5. If $AB = 640$ ft., $A'B' = 16$ in., and $A'D' = 20$ in., how far is it from A to D?

6. If $C'F'$ is measured and found to be 10 in., how far is it from C to F?

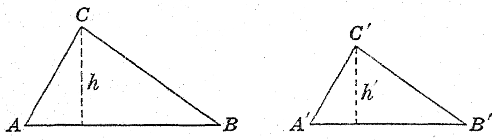
371. Ratio of the areas of similar figures. If each side of a square is 2 in., into how many squares is it divided by lines connecting the mid-points of opposite sides? What is the ratio of similitude of the large square and a small square? What is the ratio of their areas?



A triangle has a base of 6 in., and altitude 10 in. A similar triangle has a corresponding altitude of 15 in. What is the ratio of similitude? What is the base of the second triangle? What is the ratio of their areas? How does it compare with the ratio of similitude?

PROPOSITION 9. THEOREM

372. *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



Given: Triangle ABC with altitude h, and $\triangle A'B'C'$ with altitude h'; $\triangle ABC \sim \triangle A'B'C'$.

To prove: $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Plan: 1. Use the theorem in § 350.

2. Recall: "Corresponding altitudes of similar triangles . . ." (§ 306.)

Proof:

STATEMENTS	REASONS
1. Area $\triangle ABC = \frac{1}{2} h \cdot AB$, area $\triangle A'B'C' = \frac{1}{2} h' \cdot A'B'$.	1. § 348.
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{h \cdot AB}{h' \cdot A'B'}$.	2. § 350.
3. But $\dfrac{h}{h'} = \dfrac{AB}{A'B'}$.	3. § 306.
4. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	4. *Ax.* 7.

EXERCISES

In the figure for § 372:

1. If the area of $\triangle ABC = 47$ sq. in., $AB = 8$ in., and $A'B' = 12$ in., what is the area of $\triangle A'B'C'$?

2. If $AB = 2\ A'B'$, what is the ratio of the areas?

3. If $AB = 3\ A'B'$, what is the ratio of the areas?

4. The sides of a triangle are 3 in., 5 in., and 7 in. Find the sides of a similar triangle whose area is nine times as great.

5. What is the ratio of the areas of two equilateral triangles whose sides are 12 in., and 9 in., respectively?

6. If the area of an equilateral triangle is four times that of another with a side of 12 in., how long must the sides of the larger triangle be?

---

7. In $\triangle ABC$, DE is drawn parallel to BC. If $AD = 2$ in., and $DB = 3$ in., what is the ratio of the areas of the two parts into which the triangle is divided?

8. The area of a triangle is 256 sq. in. Lines parallel to the base divide one side into four equal segments. Find the areas of the four parts into which the triangle is divided.

9. The base of a triangle is 36 in. Find the length of a segment parallel to the base and drawn so that the area of the triangle above the segment is one ninth of the whole triangle.

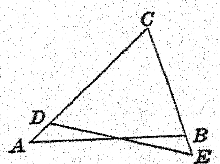
10. A line parallel to the base AB of $\triangle ABC$ and dividing $\triangle ABC$ into two equal parts cuts off what part of AC, measured from C? Compute to hundredths.

11. Prove that the areas of two similar triangles have the same ratio as the squares of corresponding altitudes.

12. Prove that the areas of two similar triangles have the same ratio as the squares of any two corresponding angle bisectors. (See Ex. 13, § 306.)

13. Prove that the areas of two similar triangles have the same ratio as the squares of any two corresponding medians. (See Ex. 7, § 317.)

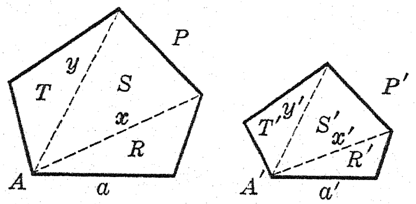
*14. Construct an isosceles triangle whose vertex angle is one of the angles of a given triangle, and whose area is equal to that of the given triangle.



SUGGESTION. — If $\triangle ABC$ is the given triangle and $\triangle CDE$ the required isosceles triangle, $\overline{CD}^2 = CA \times CB$. (See § 356.)

Proposition 10. Theorem

373. *The areas of two similar polygons have the same ratio as the squares of any two corresponding sides.*



Given: Polygons P and P' similar, with corresponding vertices A and A', and corresponding sides a and a'.

To prove: $\dfrac{P}{P'} = \dfrac{a^2}{a'^2}$.

Plan: Think: "Since $P \sim P'$ I know that $R \sim R'$, $S \sim S'$, $T \sim T'$ (§ 367). Then since the areas of $\sim$ $\triangle$ have the same ratio as the squares of corresponding sides, I can obtain a series of equal ratios."

Proof:

STATEMENTS	REASONS
1. Draw diagonals from A and A'. Then $R \sim R'$, $S \sim S'$, $T \sim T'$.	1. § 367.
2. $\dfrac{R}{R'} = \dfrac{a^2}{a'^2} = \dfrac{x^2}{x'^2} = \dfrac{S}{S'} = \dfrac{y^2}{y'^2} = \dfrac{T}{T'}$.	2. § 372.
3. $\therefore \dfrac{R}{R'} = \dfrac{S}{S'} = \dfrac{T}{T'}$.	3. *Ax.* 1.
4. $\dfrac{R+S+T}{R'+S'+T'} = \dfrac{R}{R'} = \dfrac{a^2}{a'^2}$.	4. § 364.
5. $\therefore \dfrac{P}{P'} = \dfrac{a^2}{a'^2}$.	5. *Ax.* 7.

EXERCISES

1. Two corresponding sides of two similar polygons are 5 in. and 8 in., respectively, and the area of the smaller polygon is 100 sq. in. What is the area of the larger?

2. The sides of a quadrilateral are 4 in., 6 in., 10 in., and 12 in., respectively. Find the sides of a similar quadrilateral whose area is twenty-five times as great.

3. The adjacent sides of a parallelogram are 8 ft. and 12 ft., respectively. Find the corresponding sides of a similar parallelogram whose area is one fourth as great.

4. The areas of two similar polygons are 324 sq. ft. and 576 sq. ft., respectively. What is their ratio of similitude?

5. The sum of the areas of two similar polygons is 400 sq. ft. If the ratio of similitude of the polygons is 3 : 4, what is the area of each polygon?

6. The areas of two similar hexagons are 121 sq. in. and 361 sq. in., respectively, and a side of the smaller hexagon is 6 in. Find the corresponding side of the other hexagon.

7. The side of a square is 16 in. Find the area of a square erected on its diagonal. (See Ex. 4, § 363.)

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8. The non-parallel sides of a trapezoid form with the diagonals two equal triangles.

9. If, in the parallelogram $ABCD$, point P is taken on AD and Q on CD, prove that $\triangle BPC = \triangle AQB$.

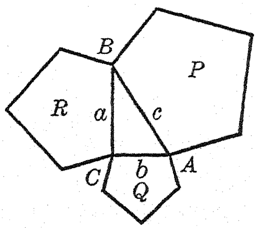
10. Prove that the areas of two similar polygons have the same ratio as the squares of corresponding diagonals.

11. Given a quadrilateral $ABCD$, construct a similar quadrilateral whose area shall be one-ninth as great as that of $ABCD$.

12. If segments are drawn connecting the middle points of the segments of the diagonals of a given trapezoid, compare the area of the trapezoid thus formed with the area of the given trapezoid.

Proposition 11. Theorem

374. A. *In any right triangle, a polygon constructed on the hypotenuse is equal in area to the sum of similar polygons constructed on the other two sides.*



Given: Right $\triangle ABC$, $\angle C$ a right angle, similar polygons P, Q, and R, with corresponding sides c, b, and a.

To prove: $P = Q + R$.

Plan: Compare Q with P and R with P, using § 373. Then add.

Proof:

STATEMENTS	REASONS
1. $\frac{Q}{P} = \frac{b^2}{c^2}$ and $\frac{R}{P} = \frac{a^2}{c^2}$.	1. § 373.
2. $\frac{Q + R}{P} = \frac{b^2 + a^2}{c^2}$.	2. *Why?*

Complete the proof.

Ex. 1. Using the method in § 374, construct an equilateral triangle equal to the sum of two given equilateral triangles.

Ex. 2. Construct a hexagon similar to two given similar hexagons and equal to their sum.

Ex. 3. Construct an equilateral triangle whose area is equal to the difference in the areas of two given equilateral triangles.

375. A. Continuity. In § 374 you have an extension of the Pythagorean theorem (§ 326).

You have seen in § 332 how §§ 328 and 329 can be considered as extensions of the same theorem from a different point of view.

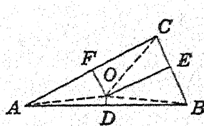
Again Ex. 5, § 360 shows that the Pythagorean theorem is a special case of the more general theorem given by Pappus (Ex. 10-12, § 347).

MISCELLANEOUS EXERCISES

1. The geometric figures of this book are printed from wax engravings. The cost of a wax engraving is computed at a certain price per square inch of its surface. If the geometric figures of this book had been made with the dimensions twice as great, how would the cost of the wax engravings have compared with the actual cost?

2. A square air duct with side 12 in. has the same capacity as how many square air ducts 3 in. on a side?

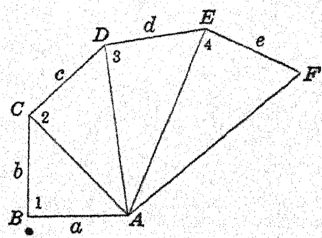
3. A trapezoid has bases of 3 in. and 1 in., respectively, and its altitude is 2 in. What is the altitude of a triangle with an equal area if the base is equal to the longer base of the trapezoid?



4. Construct the triangle in Ex. 3.

5. Point O is any point in $\triangle ABC$, $OD \perp AB$, $OE \perp BC$, $OF \perp AC$. Prove that $\overline{AD}^2 + \overline{BE}^2 + \overline{CF}^2 = \overline{DB}^2 + \overline{EC}^2 + \overline{FA}^2$.

SUGGESTION. — $\overline{AD}^2 + \overline{DO}^2 = \overline{FA}^2 + \overline{FO}^2$, etc.



***6.** Trisect a given triangle by drawing straight lines through any given point on one of the sides.

***7.** If ∠s 1, 2, 3, and 4 are right angles show that

$AF = \sqrt{a^2 + b^2 + c^2 + d^2 + e^2}$.

***8.** In the same figure, if $a = b = c = d = e = 1$, find AC, AD, AE, and AF.

***9.** Use the construction illustrated to construct the side of a square air duct the area of whose cross section is equal to the sum of the areas of five square ducts with sides 2, 3, 5, 6, and 7.

***10.** An isosceles triangle ABC has base $AB = 40$ in., and altitude $CD = 21$ in. If AC is produced to E so that $CE = 5$ in., find the area of $\triangle BCE$. (See Ex. 4, § 356.)

***11.** In $\triangle ABC$, E is taken on AC so that $3\,AE = AB$, and D is taken on AB so that $3\,AD = AC$. If $AB = 2\,AC$, compare the areas of $\triangle$s ABE and ACD. (See § 356.)

376. Summary of the Work of Unit Seven

I. ***Formulas to be memorized.***

1. *Area of a rectangle,* $A = hb$.
2. *Area of a square,* $A = a^2$, *where* a *is a side.*
3. *Area of a parallelogram,* $A = hb$.
4. *Area of a triangle,* $A = \frac{1}{2}\,hb$.
5. *Area of a trapezoid,* $A = \frac{1}{2}\,h(b + b')$.
6. *Area of a rhombus,* $A = \frac{1}{2}\,dd'$, *where* d *and* d' *are diagonals.*
7. *Area of an equilateral triangle,* $A = \frac{a^2}{4}\sqrt{3}$, *where* a *is a side.*
8. *Area of any triangle,* $A = \sqrt{s(s-a)(s-b)(s-c)}$, *where* a, b, *and* c *are sides and* s *is the semiperimeter.*
9. *The diagonal of a square with side* a, $d = a\sqrt{2}$.
10. *The altitude* h *of an equilateral triangle with side* a, $h = \frac{a}{2}\sqrt{3}$.

NOTE. — h is the number of units of length in the altitude, b the number of the same kind of units in the base, A the number of square units in the area, etc.

II. *Formulas that are not to be memorized.*

11. *In* $\triangle ABC$, *with* $\angle A$ *acute*, $a^2 = b^2 + c^2 - 2b \times p_b^c$.
12. *In* $\triangle ABC$, *with* $\angle A$ *obtuse*, $a^2 = b^2 + c^2 + 2b \times p_b^c$.
13. *In any triangle*, $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

$$h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$

NOTE. — See §§ 233 and 327 for the explanation of the symbols. Formulas 11 and 12 are from Unit VI.

III. *Important numerical relations in theorems.*

1. *In similar triangles the perimeters, corresponding altitudes, angle bisectors, or medians, have the same ratio as any two corresponding sides.*
2. *In similar polygons, the perimeters have the same ratio as any two corresponding sides.*
3. *The areas of similar triangles have the same ratio as the squares of corresponding sides; and hence as the squares of the perimeters or as the squares of corresponding altitudes, angle bisectors, or medians.*
4. *The areas of similar polygons have the same ratio as the squares of corresponding sides.*

IV. *Constructions.*

1. *To construct a triangle equal in area to a given polygon.*
2. *To construct on a given segment as side a polygon similar to a given polygon.*

REVIEW OF UNIT SEVEN

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the tests. The references given are those most closely related to the exercise.

1. Is there any difference between equal figures and congruent figures? § 342.

2. Can two rectangles have the same area and different perimeters? § 341.

3. If two parallelograms have the same area and the base of one is one third the base of the other, how do their altitudes compare? § 345.

4. Do you think you can draw a triangle with a perimeter of 3 in. that has the same area as a triangle with a perimeter of 12 in. or more? Illustrate. § 348.

5. Can a rectangle with a perimeter of 6 in. have the same area as a rectangle with a perimeter of 9 in.? With a perimeter greater than 9 in.? § 341.

6. Can two squares with different perimeters have the same area? § 341.

7. Explain two ways to find the approximate area of an irregular field with curved boundaries. See *Practical Applications*, § 359, Ex. 1, 3.

8. What is meant by the product of two segments? How can it be constructed? §§ 297, 337.

9. How can you construct the quotient of two segments? §§ 297, 337.

10. How do you find the base of a triangle if you are given the area and the altitude? § 348.

11. How do you find the altitude of a trapezoid, given the bases and the area? § 353.

12. If two triangles have equal areas, and the base of one is five times the base of the other, how do their altitudes compare? § 350.

13. The sides of one triangle are three times the sides of another triangle. How do their areas compare? § 372.

14. How can you find the area of a right triangle, one of whose acute angles is 45°, if you are given the hypotenuse? §§ 348, 360.

15. If you are given the hypotenuse of a right triangle, one of whose acute angles is 30°, how can you find the area? §§ 160, 348, 360.

For Ex. 16–21, see § 362:

16. How can you transform a triangle into a square?

17. How can you transform a triangle into a rectangle? Into a parallelogram?

18. How can you transform a square into a triangle?

19. How can you transform a rectangle into a triangle? A parallelogram into a triangle?

20. How can you transform any polygon into a triangle?

21. How can you transform any polygon into a square? Into a parallelogram?

22. Explain two ways of constructing $\sqrt{2}$. §§ 337, 360.

23. How can you construct a square equal to the sum of two given squares? § 362, Ex. 4.

24. How can you construct a square equal to the difference of two given squares? § 362, Ex. 5.

Complete the following:

25. The perimeters of two similar polygons ... § 365.

26. The areas of two similar polygons ... § 373.

27. A square erected on a given segment is —— times the square erected on half the segment. § 373.

28. The areas of two triangles are to each other as ... § 350.

29. Two triangles having equal altitudes are to each other ... § 351.

30. Two parallelograms having equal bases are to each other as ... § 347.

31. The area of a rhombus is ... § 363, Ex. 2.

32. The area of an equilateral triangle is ... § 363, Ex. 5.

33. A median of a triangle divides the triangle into two —— triangles. § 351.

34. If each of the sides of a triangle is doubled, the area is ——. § 372.

35. What does the s in Hero's formula for the area of a triangle mean? § 358.

36. If one triangle has a perimeter four times that of another similar triangle, how do the areas compare? § 372.

37. The altitude of an equilateral triangle is one-third that of another equilateral triangle. The area of the first triangle is one —— that of the second. § 372.

38. If the altitude of an isosceles triangle is doubled, the base remaining the same, can you tell how the area is affected? § 352.

NUMERICAL EXERCISES

See the formulas in § 376.

1. What is the area of a parallelogram with sides 10 in. and 8 in. and included angle 30°?

2. What is the area of a rhombus with side 10 in., if an angle is 45°?

3. What is the area of the rhombus in Ex. 2 if an angle is 30°? If an angle is 60°? 120°?

4. Find the area of an isosceles right triangle whose legs are 20 in.

5. One of the bases of a trapezoid is 15 ft. and its altitude is 6 ft. If the area is 75 sq. ft., find the other base.

6. Find the side of a square equal to the sum of three squares whose sides are 8 in., 9 in., and 12 in.

7. Find the area of a triangle with sides 51 in., 75 in., and 78 in.

8. Two similar triangles have two corresponding sides 6 in. and 15 in., respectively. The larger triangle has how many times the area of the smaller?

9. Find the area of an isosceles triangle if the base is 16 ft. and one of the legs is 17 ft.

10. One side of an equilateral triangle is 10 ft. Find the area.

11. Find the side and area of an equilateral triangle whose altitude is 20 in.

12. The diagonal of a rectangle is 37 in. Find its area if one side is 12 in.

13. The diagonal of a square is 15 in. Find its area.

14. The area of a rhombus is 240 sq. ft. If one diagonal is 60 ft., what is the other?

15. One diagonal of a rectangle is 20 in., and it makes an angle of 30° with one side. Find the area of the rectangle.

16. The areas of two similar polygons are 160 sq. ft. and 360 sq. ft. If a side of the smaller is 10 ft., what is the corresponding side of the other?

17. Two similar polygons have areas 250 sq. in. and 160 sq. in. If a side of the first is 6 in., find the corresponding side of the second.

18. Find the side of an equilateral triangle whose area is $25\sqrt{3}$ sq. in.

19. The areas of two similar triangles are 200 sq. in. and 450 sq. in. If an altitude of the first is 25 in., find the corresponding altitude of the second.

20. Two chords AB and CD intersect at O. If AB is 24 ft., AO is 6 ft., and CO is 9 ft., find OD.

21. A tangent to a circle from an external point is 12 in. If the external segment of a secant from the same point is 8 in., what is the length of the whole secant?

22. If, in Ex. 21, the length of another secant from the same point is 16 in., what is the length of the external segment?

23. The sides of a triangle are 3 in., 5 in., and 6 in. The longest side of a similar triangle is 15 in. Compute the other sides.

24. If the radius of a circle is 10 in., find the length of a chord whose arc is 60°.

25. In Ex. 24, if the arc is 90°, how long is its chord?

26. If the arc is 30°, how long is its chord?

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27. Find the altitude on the 51 in. side of the triangle in Ex. 7. (See formula 13, § 376.)

28. The area of a rhombus is 75 sq. ft. If the ratio of the diagonals is 2 : 3, find the diagonals.

29. The sum of the areas of two similar triangles is 225 sq. ft. If one area is $1\frac{7}{8}$ times the other, what is the ratio of similitude of the triangles?

30. Find the area of a rhombus if its perimeter is 40 ft. and one diagonal is 16 ft.

31. Two sides of a triangle are 12 in. and 16 in. If the included angle is 60°, what is its area?

32. What is the area of the triangle formed by the radii and chord, in Ex. 24, if the included angle is 30°? 45°?

33. The perimeter of a quadrilateral is 46 in. If the radius of the inscribed circle is 6 in., what is its area? (See Ex. 10, § 363.)

34. Two triangles each have an angle of 70°. If the including sides are 16 in. and 12 in. for the first and 15 in. and 24 in. for the second, what is the ratio of their areas? (See § 356.)

35. A point P lies outside a circle at a distance of 13 in. from the center. A secant from P cuts the circle at Q and R so that $PQ = 9$ in. and $QR = 7$ in. Find the radius of the circle.

36. In Ex. 35 what is the length of the tangent from P?

37. If the sides of a right triangle are in the ratio of 4 : 5, find the ratio of the parts into which the altitude on the hypotenuse divides the hypotenuse.

38. The sides of a triangle are in the ratio 15 : 13 : 14. If the area of the triangle is 336 sq. in., what are its sides?

39. The bases of a trapezoid are 5 ft. and 20 ft. and its non-parallel sides are 12 ft. and 9 ft. Find its area.

40. The altitude of a triangle is 15 in. How far from the vertex must a parallel to the base be drawn to cut off a triangle equal to one fourth of the original triangle?

***41.** If one leg of a right triangle is 25 in. and the altitude on the hypotenuse is 7 in., what is its area?

***42.** If the diagonal of a rectangle is 58 ft. and one side is 42 ft., find the side of an equilateral triangle having the same area.

GENERAL EXERCISES

1. The area of a square is half the square on its diagonal.

2. The diagonals of a trapezoid divide the trapezoid into two pairs of triangles, one pair similar and one pair equal. (See Ex. 6, § 317.)

3. In $\triangle ABC$, if AD is an altitude, prove $\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{DC}^2$.

4. The altitude to the hypotenuse of a right triangle divides it into two triangles whose areas have the same ratio as the squares of the legs of the given triangle.

5. In $\triangle ABC$, $\angle A$ is 30°. Prove that the area of $\triangle ABC$ is $\frac{1}{4}$ $AB \times AC$.

6. The area of an isosceles right triangle is one-fourth the area of the square on the hypotenuse.

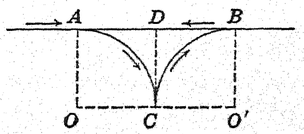
---

7. If P is any point within a parallelogram $ABCD$, then $\triangle PAD + \triangle PBC$ is half the parallelogram.

8. If a triangle ABC is formed by the intersection of three tangents to a circle, two of which, AD and AE, are fixed, while the third, BC, touches the circle at a variable point F on the arc DE, prove that the perimeter of the triangle ABC is constant and equal to $AD + AE$.

9. Construct an equilateral triangle equal to a given triangle.

10. Construct an equilateral triangle equal to a given polygon.

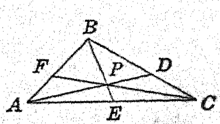


***11.** A railroad Y consists of three tracks, AC, CB, and AB, upon which a train is reversed in direction, by moving as shown by the arrows, backing from C to B. In constructing a railroad Y, AB is a straight track, and point A is given. It is required to locate the curves AC and CB with given radii R and r, respectively.

Prove that $AD = DC = DB = \sqrt{R \times r}$.

SUGGESTION. — Draw OD and $O'D$. Prove $\triangle OCD \sim \triangle O'CD$.

***12.** If, through any point P within a triangle, lines AD, BE, and CF are drawn, then $\triangle APC : \triangle APB = DC : BD$.



SUGGESTION. — Compare $\triangle$s PDC and PBD, using the common base PD; then compare them, using the common altitude from P. Also compare $\triangle$s APB and APC.

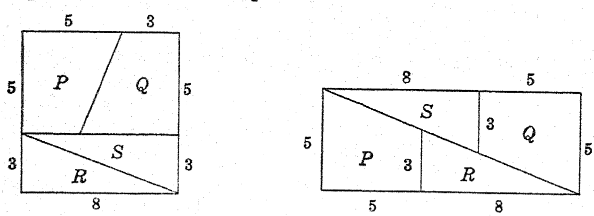
***13.** The lines AD, BE, and CF are drawn from the vertices of $\triangle ABC$ so that they meet at point P. Show that

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$$

SUGGESTION. — Use the result of Ex. 12.

***14.** If P is any point on diagonal AC produced of parallelogram $ABCD$, then $\triangle PCD = \triangle PCB$.

***15.** Explain the fallacy in the following puzzle: A piece of paper 8 in. square is cut into four pieces P, Q, R, and S, as shown in the figure. These pieces are then placed in new positions so as to form, apparently, a rectangle whose area is 65 sq. in.

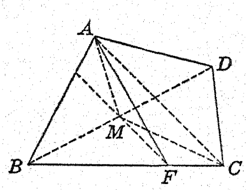


SUGGESTION. — Use similar triangles. If the triangles fitted together so that their sides formed a true diagonal of a rectangle, what proportion would follow?

***16.** Construct a polygon similar to one of two given polygons and equal to the other.

ANALYSIS. — If it is required to construct a polygon similar to P and equal to Q, let a and b be sides of squares equal to P and Q, respectively. Let R be the required polygon, and r a side of R corresponding to side p of P. Then $\frac{P}{R} = \frac{p^2}{r^2}$. Why? Also $\frac{P}{Q} = \frac{a^2}{b^2}$. Why? Hence, since $Q = R$, $\frac{a^2}{b^2} = \frac{p^2}{r^2}$. Why? Thus r is the fourth proportional to a, b, and p.

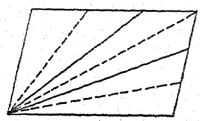
***17.** Bisect any quadrilateral by a straight line drawn through one of the vertices.



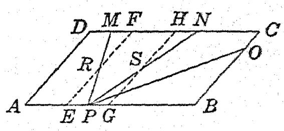
ANALYSIS. — Let A be the vertex. If M is the middle point of BD, then AM and CM divide the quadrilateral into two equal parts. Hence it remains to construct a triangle equal to $\triangle AMC$, having for base AC and a vertex in one of the sides of the quadrilateral.

***18.** Trisect any given quadrilateral by straight lines drawn through any vertex.

HINT. — Trisect a diagonal and proceed as in Ex. 17.



***19.** Divide a given parallelogram into any required number of equal parts by straight lines drawn from one of its vertices.



***20.** Divide a given parallelogram into four equal parts by drawing straight lines through any given point on one of its sides.

SUGGESTION. — Let P be the given point. Divide AB into four equal parts. Draw EF and GH parallel to AD. Bisect EF at R and GH at S. Then $APMD = PMN = \frac{1}{4} ABCD$. Having drawn PM and PN, construct PO by Ex. 17.

PRACTICE TESTS

These are practice tests. See if you can do all the exercises correctly without referring to the test. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT SEVEN

TEST ONE

Numerical Exercises

1. The area of a triangle is 30 sq. in. Its altitude is 10 in. What is its base? § 348.

2. A rectangle is 30 in. wide and 50 in. long. What is the area of a rectangle just as wide but only four-fifths as long? § 341.

3. Two parallelograms have bases each equal to 8 in. What is the ratio of their areas if the altitude of one is 7 in. and the altitude of the other 14 in.? § 347.

4. A rectangular field is 80 rd. long and 20 rd. wide. How much fencing would a square field require if it had the same area? § 341.

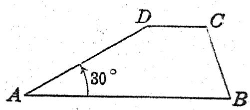
5. The area of a trapezoid is 36 sq. in. One base is double the other, and the altitude is 4 in. Find the bases. § 353.

6. A square has a side of 6 in. A triangle has a base of 9 in. What is the altitude of the triangle, if its area is equal to that of the square? §§ 341, 348.

7. $\triangle ABC \sim \triangle A'B'C'$. $AB = 9$ in. $A'B'$, the side of $\triangle A'B'C'$ corresponding to AB, is 12 in. If the area of $\triangle ABC$ is 36 sq. in., find the area of $\triangle A'B'C'$. § 372.

8. A side of a square is 6 in. Find its diagonal to the nearest tenth. § 363.

9. $\triangle ABC \sim \triangle A'B'C'$. AB is three times as large as the corresponding side $A'B'$. What is the ratio of their areas? § 372.



10. Find the area of trapezoid $ABCD$, if $AB = 12$ in., $DC = 8$ in., $AD = 10$ in., and angle A is 30°. § 353.

11. Two similar pentagons have corresponding sides AB and $A'B'$ equal to 6 in. and 8 in., respectively. What is the ratio of their perimeters? § 365.

12. The area of a polygon is 64 sq. in. The area of a similar polygon is 40 sq. in. If the smallest side of the first is 8 in., what is the smallest side of the second? § 373.

TEST TWO

True-False Statements

If a statement is always true, mark it so. If it is not, replace each word in italics by a word which will make it a true statement.

1. Equal polygons are *always* congruent. § 342.

2. Two parallelograms having equal altitudes are to each other as their *bases*. § 346.

3. The area of a trapezoid is equal to the product of its altitude by *half* the segment connecting the mid-points of the legs. § 354.

4. A square and a rectangle have the same perimeter. The square has the *larger* area. § 342.

5. The area of a *triangle* is equal to the product of its base by its altitude. § 348.

6. The perimeters of two similar polygons have the same ratio as *any two* sides. § 365.

7. If a triangle has a base *double* that of a square, and their areas are the same, the altitude of the triangle is the same as that of the square. §§ 341, 348.

8. A trapezoid is equal in area to a parallelogram if the *median* of the trapezoid equals the base of the parallelogram, and the altitudes are equal. § §343, 353.

9. The area of a right triangle whose legs are a and b is *twice* ab. § 363.

10. If r is the radius of a circle inscribed in an equilateral triangle, the altitude of the triangle is *three* times r. § 363.

11. A triangle is equal to *half* a parallelogram having the same base and altitude. §§ 341, 348.

12. The areas of two similar triangles have the same ratio as the squares of their *perimeters*. § 376–III.

TEST THREE

Drawing Conclusions

Give a conclusion that may be drawn from the hypothesis stated. The conclusion should be about areas in all except Ex. 4 and 12.

1. Triangle ABC has sides a, b, and c. § 358.

2. Polygon $P \sim$ polygon P'. P has side a, and P' has corresponding side a'. § 373.

3. Triangle ABC has base b, and altitude h. $\triangle DEF$ has base b, and altitude h'. § 352.

4. Polygon $ABCDE \sim$ polygon $A'B'C'D'E'$ and the lettering is counterclockwise in each. Diagonals from A and A' divide $ABCDE$ into triangles ABC, ACD, ADE, and diagonals from A' divide $A'B'C'D'E'$ into corresponding triangles $A'B'C'$, $A'C'D'$, $A'D'E'$. § 367.

5. Triangle $ABC \sim \triangle A'B'C'$. h is the altitude drawn from C, and h' the altitude drawn from corresponding point C'. § 376–III.

6. In $\triangle ABC$, $\angle A = 40°$. In $\triangle DEF$, $\angle D = 40°$. § 356.

7. Parallelogram P and triangle T have their bases and altitudes equal respectively. §§ 343, 348.

8. In $\triangle ABC$, $\angle C = 90°$. P, Q, and R are similar polygons having AB, CA, and BC as corresponding sides, respectively. § 374.

9. Trapezoid T has median m and altitude h. Parallelogram P has base m and altitude h. §§ 343, 354.

10. Parallelogram P has base b and altitude h. Parallelogram Q has base b' and altitude h'. § 345.

11. A is the area and p the perimeter of polygon P. A' is the area and p' the perimeter of polygon P'. $P \sim P'$. § 376–III.

12. $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = r$. § 364.

CUMULATIVE TESTS ON THE FIRST SEVEN UNITS

TEST FOUR

Numerical Exercises

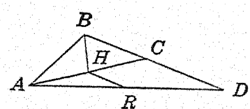
1. How many degrees in the angle formed by the bisectors of two adjacent complementary angles? § 38.

2. Two sides of an inscribed triangle have arcs which are one-fifth and one-eighth of the circle, respectively. Find the angles of the triangle. § 241.

3. If two angles of a triangle are 45° and 75°, respectively, how many degrees in the complement of the angle at the third vertex? § 123.

4. How many sides has a polygon if the sum of the exterior angles equals the sum of the interior angles? §§ 133, 134.

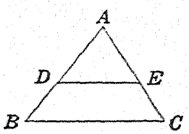
5. Triangles ABH and BHC are congruent. AC is a straight line and R is the mid-point of AD. If CD is 10 in., find HR. § 302.



6. From an external point P a tangent PA is drawn to a circle and also a secant passing through the center of the circle. If PA is 6 in. and the external segment of the secant is 4 in., find the radius of the circle. § 335.

7. The hypotenuse AB of a right triangle is 18 in. If segment BD made by the altitude CD is 8 in., find BC. § 323.

8. The altitude of a triangle is 3 times the base, and its area is $37\frac{1}{2}$ sq. yd. Find the length of the base and altitude. § 348.



9. In the triangle ABC, DE is parallel to BC. If AD is 8 in., AB is 12 in., and DE is 10 in., find BC. § 302.

10. A square and a rectangle have equal perimeters, 144 ft., and the length of the rectangle is five times its breadth. Compare the area of the square and rectangle. § 341.

11. Find the length of the longest and shortest chord that can be drawn through a point 12 inches from the center of a circle whose radius is 20 inches. § 210, see Ex. 10.

12. The areas of two similar triangles are 25 sq. ft. and 144 sq. ft. Find the ratio of their corresponding sides. § 372.

TEST FIVE

True-False Statements

If a statement is always true, mark it so. If it is not, replace each word in italics by a word which will make it a true statement.

1. The area of a trapezoid is equal to the product of the line joining the mid-points of its non-parallel sides by its *altitude.* § 354.

2. If the opposite sides of a quadrilateral are equal the figure is a *rectangle.* § 146.

3. Two polygons may be mutually *equilateral* without being similar. § 302.

4. If a triangle has sides 5 in., 12 in., and 13 in., it is a *right* triangle. § 326.

5. If a regular polygon has n sides each interior angle is $\frac{360°}{n}$. § 133.

6. The perpendicular bisector of a segment is determined by *two* points equidistant from the extremities of the segment. § 87.

7. Either leg of a right triangle is a mean proportional between the *hypotenuse* and one of the segments made by the altitude on the hypotenuse. § 323.

8. Equal *complementary* angles are each 45°. § 38.

9. If, from a point outside a circle, a secant and a tangent are drawn, the tangent is a *third* proportional between the whole secant and the external segment. § 335.

10. In an equilateral triangle the radius of the inscribed circle is *half* the radius of the circumscribed circle. § 363, Ex. 6, 7.

11. If two *angles* are inscribed in the same arc they are equal. § 242.

12. Two triangles having an angle of one equal to an angle of the other are to each other as the *products* of the sides including the equal angles. § 356.

TEST SIX

Matching Exercises

In group B brief descriptions of the terms in group A are given. Match them correctly.

A	B
I. Similar polygons § 284	1. A polygon with four sides.
II. Trapezoid § 140	2. A fraction.
III. Perpendicular § 26	3. Two angles whose sum is a straight angle.
IV. Third proportional § 290	4. Polygons that have the same shape.
V. Ratio § 285	5. A part of a circle less than a semicircle.
VI. Supplementary angles § 38	6. A line cutting a circle.
VII. Quadrilateral § 131	7. A quadrilateral with only one pair of parallel sides.
VIII. Vertical angles § 45	8. An angle formed by chords intersecting on a circle.
IX. Minor arc § 187	9. The last term in a proportion involving three terms.
X. Secant § 211	10. Equal angles formed by intersecting lines.
XI. Proportion § 286	11. A line making right angles with another line.
XII. Inscribed angle § 239	12. An equality between two fractions.

TEST SEVEN

Numerical Exercises

1. Two angles of a triangle are 44° and 36°, respectively. How large is the exterior angle at the third vertex? § 128.

2. Chords AB and CD intersect at P and form an angle of 18°. If arc AC is 24°, find arc BD. § 333.

3. Each interior angle of a polygon is 144°. How many sides has the polygon? § 133.

4. The hypotenuse AB of right triangle ABC is 41 rd. If side AC is 40 rd., how long is side BC? § 326.

5. In triangle ABC, angle A is 70°, AB is 20 ft., AC is 30 ft., and the area is 282 sq. ft. Find the area of triangle DEF if angle D is 70°, DE is 8 ft., and DF is 25 ft. § 356.

6. Two secants intersect at point P and intercept arcs AC and BD on a circle. If angle P is 36° and arc AC is 8°, how many degrees in arc BD? § 248.

7. The median of a trapezoid is 10 in., and one of the bases is 16 in. How long is the other base? § 157.

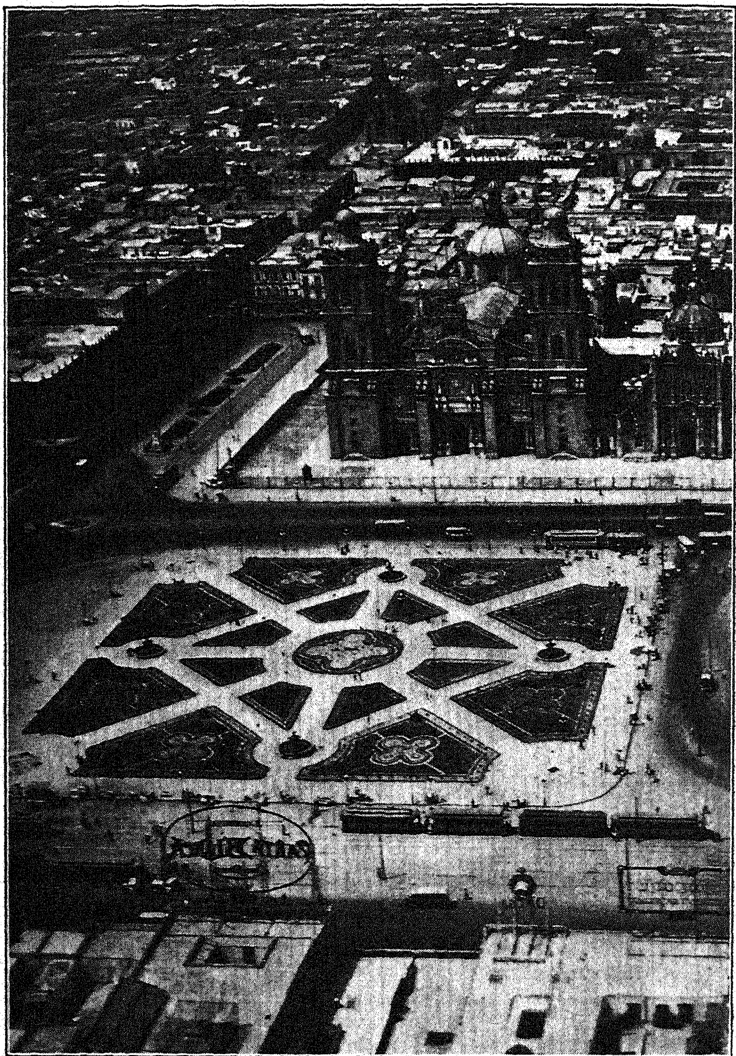
8. In parallelogram $ABCD$, angle B is 150°. If side BC is 48 ft., how long is the altitude from C to AB? § 160.

9. Chords AB and CD intersect at point P. If AB is 15 in., CP is 1 in., and PD is 36 in., how long is AP? § 333.

10. In triangle ABC, angle C is 90° and CD is perpendicular to AB. If CD is 15 in., and AD is 9 in., how long is AB? § 323.

11. The sum of the areas of two similar triangles is 169 sq. in. If the ratio of similitude is 5 : 12, what is the area of the larger triangle? § 372.

12. Chord BC is 12 in. and the perpendicular CD drawn to diameter AB is $\sqrt{108}$ in. What is the diameter of the circle?



GEOMETRY IN LANDSCAPING © *E. P. Jones*

The famous Zocalo Gardens in Mexico City show an unusual use of geometric design.

UNIT EIGHT

REGULAR POLYGONS

377. In this unit you will make a study of regular polygons and of their relation to the circle. You will recall that *a regular polygon is a polygon whose sides are all equal and whose angles are all equal.* Thus an equilateral triangle is a regular polygon and so also is a square. Why is a rhombus not a regular polygon? The following exercises will review what you have learned about regular polygons.

EXERCISES

1. How many degrees are there in each angle of a regular pentagon? Hexagon? Octagon? (See § 133.)

2. How many degrees in each exterior angle of a regular polygon, if each interior angle is 120°? 108°? 135°?

3. How many sides has a regular polygon if each interior angle is 168°? 160°? 179°?

4. Make a table giving (*a*) the sum of the interior angles; (*b*) each interior angle; (*c*) each exterior angle; (*d*) the sum of the exterior angles; for regular polygons of 6, 8, 10, 12, 15, and 20 sides.

5. How is a central angle measured? An inscribed angle? The angle made by a tangent and a chord? An angle made by two tangents? By a tangent and a secant? By two secants?

6. When is a polygon inscribed in a circle? Circumscribed about a circle?

7. Do you think that a circle can be circumscribed about *any* polygon?

8. Tell how to inscribe a circle in an equilateral triangle.

9. Tell how to circumscribe a circle about an equilateral triangle.

10. If the side of an equilateral triangle is a, what is its altitude? What is the radius of the inscribed circle? Of the circumscribed circle?

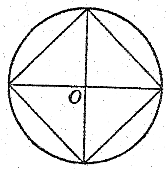
11. Two perpendicular diameters are drawn in a circle. Prove that the figure formed by connecting their ends is a square.

12. Can you prove that a chord equal in length to a radius has a central angle of 60°, and hence is a side of a hexagon inscribed in a circle?

13. What must be proved to show that the hexagon in Ex. 12 is a regular hexagon?

Construction XXIII

378. *To inscribe a square in a given circle.*



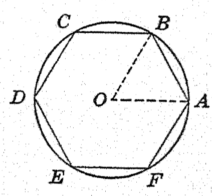
Given: Circle O.

Required: Inscribe a square in circle O.

Write out the construction and proof. (See Ex. 11.)

Construction XXIV

379. *To inscribe a regular hexagon in a given circle.*



Given: Circle O.

Required: Inscribe a regular hexagon in circle O.

Write the construction and proof in full. (See Ex. 12 and 13.)

EXERCISES

1. If the side of a regular hexagon inscribed in a circle is a, what is the radius of the circumscribed circle?

2. If the side of a regular hexagon is a, what is its area?

Hint. — There are six equilateral triangles. The area of each is $\frac{a^2}{4}\sqrt{3}$.

3. Prove that, if all the diagonals are drawn from any vertex of a regular hexagon inscribed in a circle, they divide the angle at that vertex into equal parts.

4. If R is the radius of the circle, prove that the side of an inscribed square is $R\sqrt{2}$.

5. If R is the radius of the circle, prove that the area of the inscribed square is $2R^2$.

6. Prove that the point of intersection of the diagonals of a square is equidistant from the sides of the square.

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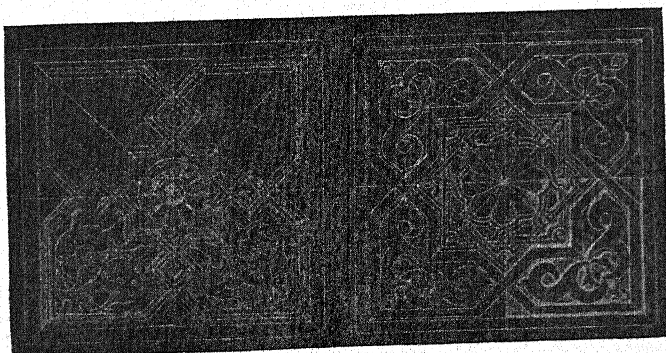
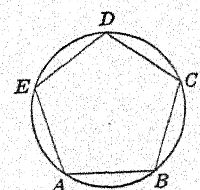
7. From the result of Ex. 6, show how to inscribe a circle in a square.

8. How can you inscribe a circle in the regular hexagon constructed in § 379?



DESIGN FOR METAL CEILING

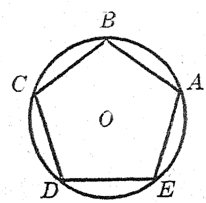
380. A circle divided into equal arcs. In § 378 we have found that if a circle is divided into four equal parts and the points of division are connected, the resulting figure is a regular polygon. In § 379 we found the same result true when the circle is divided into six equal arcs.



If a circle is divided into any number (more than two) of equal arcs, can you prove that the figure formed by connecting the points of division is a regular polygon? What must be proved to show that a figure is a regular polygon?

PROPOSITION 1. THEOREM

381. *If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon.*



Given: Circle O divided into equal arcs at A, B, C, D, and E.

To prove: $ABCDE$ is a regular polygon.

Plan: Mentally: "To prove that $ABCDE$ is a regular polygon I must prove that it is equilateral and equiangular."

Proof:

STATEMENTS	REASONS
1. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$.	1. *Why?*
2. $AB = BC = CD = DE = EA$.	2. *Why?*
3. $\angle A$ has the same measure as $\frac{1}{2}$ arc $BCDE$.	3. *Why?*
$\angle B$ has the same measure as . . .	

Complete the proof in full.

382. COROLLARY 1. *An equilateral polygon inscribed in a circle is a regular polygon.*

383. COROLLARY 2. *If lines are drawn from the mid-point of each arc determined by a side of a regular inscribed polygon to its extremities, a regular inscribed polygon of double the number of sides is formed.*

384. Corollary 3. *Regular inscribed polygons of 4, 8, 16, 32, etc., sides can be constructed.* (§ 378.)

385. Corollary 4. *Regular inscribed polygons of 6, 12, 24, 48, etc., sides can be constructed; and, by joining the alternate vertices of an inscribed hexagon, an inscribed equilateral triangle is formed.* (§ 379.)

EXERCISES

1. Draw a circle with radius 2 in. and in it inscribe a regular octagon.

2. Prove that the perimeter of the octagon constructed in Ex. 1 is greater than the perimeter of the square inscribed in the same circle.

3. Draw a circle and in it inscribe an equilateral triangle.

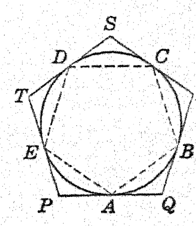
4. Draw a circle with radius 2 in. and in it inscribe a regular polygon with 12 sides.

5. In the construction of Ex. 4 connect every fifth vertex, beginning with any one, and thus form a 12-pointed star. What is the sum of the angles in the points of the star?

6. Divide a circle into eight equal parts and form an eight-pointed star by joining every third point of division. Find the sum of the angles in the points of the star.

***7.** Prove that the perimeter of a regular inscribed polygon is less than that of an inscribed polygon with twice as many sides.

386. Tangents at the points dividing a circle into equal arcs form a regular polygon.



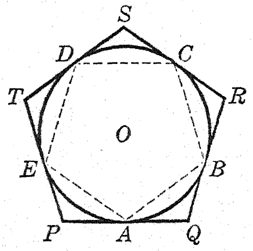
What is the measure of $\angle$s P, Q, R, S, and T? Why then are they equal?

What kind of triangles are formed by connecting the points A, B, C, D, and E? See if you can prove that they are congruent.

Can you give the proof in full that $PQRST$ is a regular polygon?

Proposition 2. Theorem

387. *If a circle is divided into equal arcs, the tangents at the points of division form a regular circumscribed polygon.*



Given: Circle O divided into equal arcs at A, B, C, D, and E; tangents PQ, QR, etc., at A, B, etc.

To prove: $PQRST$ is a regular polygon.

Plan: Prove $PQRST$ equiangular and equilateral.

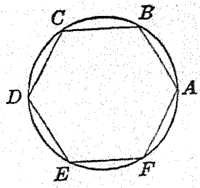
Proof:

STATEMENTS	REASONS
1. Connect points A, B, C, D, E. $\triangle$ APE, BQA, etc., are isosceles.	1. *Why?*
2. $\angle$ P, Q, R, etc., are equal.	2. *Prove in full* (§ 248).
3. $\triangle$ APE, BQA, etc., are congruent.	3. *Prove.*
4. $AQ = QB = BR =$ etc.	4. *Prove.*
5. $PQ = QR = RS =$ etc.	5. *Why?*

Write the proof in full.

388. Corollary. *If tangents are drawn at the midpoints of the arcs of adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.*

EXERCISES



1. In a regular hexagon $ABCDEF$ inscribed in a circle prove that $\angle DCA = \angle AED$.

2. In the hexagon of Ex. 1, two sides AB and EF, when produced intersect at P. Prove that $PA \times PB = PF \times PE$.

HINT. — See § 336.

3. In the hexagon of Ex. 1 prove that diagonals AD, BE, and CF are diameters.

4. A square having a side of 2 in. is inscribed in a circle. What is the radius of the circle?

5. If a square is inscribed in a circle, its diagonal is a diameter of the circle.

6. The area of a square constructed on a diameter of a circle is equal to twice the area of the square inscribed in the circle.

---

7. Prove that the perimeter of any regular circumscribed polygon is greater than that of a regular circumscribed polygon of twice as many sides.

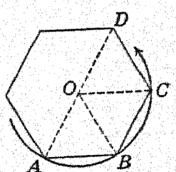
8. Construct a square whose diagonal is 3 in. (See § 378.)

9. Circumscribe a regular octagon about a circle of 2 in. radius.

10. Prove that the perimeter of the octagon in Ex. 9 is less than the perimeter of the square circumscribed about the same circle.

***11.** Find the side of the regular octagon inscribed in a circle of radius 2 in.

389. Circumscribing a circle about a regular polygon.



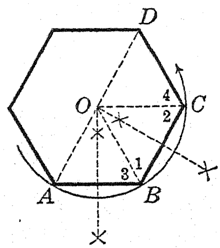
You know that a circle can always be drawn through any three points A, B, and C not in a straight line. How do you find the center O of this circle (§ 184)?

If the circle also passes through point D, how will OD compare with OA, OB, and OC?

If OA, OB, OC, and OD are drawn, what kind of triangles are AOB and BOC? See if you can prove that the circle passes through D by showing that $\triangle AOB \cong \triangle COD$.

Proposition 3. Theorem

390. A *circle can be circumscribed about any regular polygon.*



Given: Regular polygon $ABCD \ldots$

To prove: A circle may be circumscribed about $ABCD \ldots$

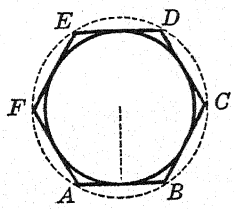
Plan: Think: "I can construct a ⊙ through the three points A, B, and C (§ 184). Then if I can prove that $OA = OD$, I will know the ⊙ passes through D."

Proof:

STATEMENTS	REASONS
1. A ⊙ can be drawn through A, B, and C. Let its center be O. Draw OA, OB, OC, and OD.	1. § 184.
2. $\angle ABC = \angle BCD, AB = CD$.	2. § 135.
3. Since $OB = OC$, $\angle 1 = \angle 2$.	3. § 69.
4. $\therefore \angle 3 = \angle 4$.	4. *Ax.* 3.
5. $\triangle ABO \cong \triangle COD$.	5. § 64.
6. $OA = OD$.	6. § 58.
7. Hence a ⊙ with O as center and OA as radius will pass through D. It can be shown that the ⊙ will pass through the other vertices.	7. *Post.* 13.

PROPOSITION 4. THEOREM

391. *A circle can be inscribed in any regular polygon.*



Given: Regular polygon $ABCDEF$.

To prove: A circle can be inscribed in $ABCDEF$.

Plan: "If a circle were circumscribed about the polygon, the sides of the polygon would be equal chords in the circle." See if you can complete the proof. Recall § 203.

Proof: For you to write out.

EXERCISES

1. If the radius of a circle inscribed in an equilateral triangle is 3 in., what is the altitude of the triangle?

2. If the radius of a circle circumscribed about an equilateral triangle is 15 in., what is the altitude of the triangle?

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3. The side s of an equilateral triangle in terms of its altitude is $s = \frac{2h}{3}\sqrt{3}$, and the area A is $A = \frac{h^2}{3}\sqrt{3}$. Find the side of an equilateral triangle if the radius of the inscribed circle is 3 in. Find the area.

4. Find the side and area of an equilateral triangle, if the radius of the circumscribed circle is 15 in.

***5.** Find the side of a regular hexagon if the radius of the inscribed circle is 12 in.

392. The apothem of a regular polygon is the radius of the inscribed circle, and the **radius** of a regular polygon is the radius of the circumscribed circle.

The common center of the circles is called the **center,** and the central angle whose chord is a side of the regular polygon is called the **central angle.**

Theorem. *Each central angle of a regular polygon of n sides is* $\frac{360°}{n}$.

393. Angles of a regular polygon. You know that the sum of the angles of a polygon having n sides is $(n - 2)$ straight angles. Hence,

Theorem. *Each angle of a regular polygon of n sides is* $\frac{n-2}{n}$ *straight angles.*

394. Perimeter of a regular polygon. Since the sides of a regular polygon are equal, if each side is s, and there are n sides,

Theorem. *The perimeter of a regular polygon is ns.*

EXERCISES

1. How large is the central angle of an equilateral triangle? Of a square? Of a regular polygon with 12 sides?

2. If the altitude of an equilateral triangle is 12 in. find its radius and apothem.

3. If the side of a square is 20 in., find its radius.

---

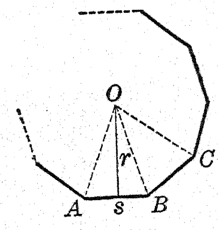
4. If the side of an equilateral triangle is 6 in., find its radius and apothem.

5. If the radius of a circle is 4 in., find the area of an inscribed equilateral triangle. (See Ex. 3, § 391.)

*6. If the area of an equilateral triangle is $12\sqrt{3}$ sq. in., find its radius.

Proposition 5. Theorem

395. *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*



Given: s the side, p the perimeter, r the apothem, and O the center of regular polygon $ABC\ldots$ having n sides.

To prove: $ABC\ldots. = \frac{1}{2}rp.$

Plan: Draw radii and find the areas of the ▲ thus formed.

Proof:

STATEMENTS	REASONS
1. Draw OA, OB, $OC\ldots$ thus forming n ▲.	1. *The polygon has n sides.*
2. The altitude of each triangle is r and its side s.	2. *Why?*
3. The area of each $\triangle = \frac{1}{2}rs$.	3. § 348.
4. The area of n ▲ $= \frac{1}{2}rns$.	4. *Ax.* 4.
5. But $ns = p$.	5. § 394.
6. $\therefore ABC\ldots = \frac{1}{2}rp$.	6. *Ax.* 7.

Ex. 1. The radius of a circle is 6 in. Find the area of the regular hexagon inscribed in the circle.

Hint. — See Ex. 2, § 379.

Ex. 2. What is the area of a square circumscribed about a circle whose radius is 10 in.?

EXERCISES

1. Compare the central angle of an equilateral triangle with an exterior angle at any vertex. Do the same for a square. For a regular pentagon. For a regular hexagon.

2. The apothem of an equilateral triangle is 10 in. What is its radius?

3. A side of an equilateral triangle inscribed in a circle is 3 in. Find the side of the equilateral triangle circumscribed about the same circle.

4. Find the ratio of the areas of the triangles in Ex. 3.

5. The radius of a circle is 2 in. Show that the area of the regular circumscribed hexagon is 13.86– sq. in.

---

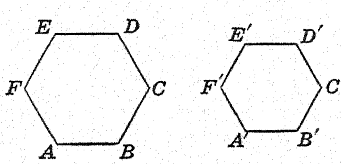
6. Prove that the central angle of a regular polygon is the supplement of an angle of the polygon.

7. Prove that the apothem of an equilateral triangle is one half the radius.

8. Prove that the side of the equilateral triangle circumscribed about a circle is twice the side of the equilateral triangle inscribed in the circle.

9. The radius and apothem of an equilateral triangle inscribed in a circle are one-half the radius and apothem of the circumscribed triangle.

*10. An equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.



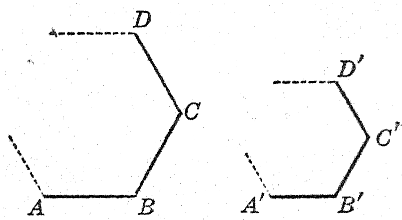
396. Regular polygons with the same number of sides are similar. Can you prove it? What two things must be proved to show that the polygons are similar?

How large is each angle of a regular polygon having n sides (§ 393)?

Why is $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$ etc.?

PROPOSITION 6. THEOREM

397. *Regular polygons of the same number of sides are similar.*



Given: Regular polygons AD and $A'D'$, each having n sides.

To prove: $AD \sim A'D'$.

Plan: Prove that the angles are equal and the sides proportional.

Proof:

STATEMENTS	REASONS
1. Each $\angle$ of AD and $A'D' = \frac{n-2}{n}$ st. $\angle$s.	1. § 393.
2. Hence $\angle A = \angle A'$, $\angle B = \angle B'$, etc.	2. *Ax.* 1.
3. $AB = BC = CD$, etc., and $A'B' = B'C' = C'D'$, etc.	3. § 135.
4. Hence $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} =$ etc.	4. *Ax.* 5.
5. $\therefore AD \sim A'D'$.	5. § 302.

Ex. 1. A regular polygon is circumscribed about a circle O. Two adjacent sides are tangent to the circle at points P and Q, respectively. Compare the angle at any vertex with the angle POQ if the regular polygon is a triangle; a hexagon; an octagon.

Ex. 2. Find the side of a regular polygon of 12 sides inscribed in a circle whose radius is 2 in.

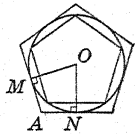
EXERCISES

1. The perimeters of two regular polygons having the same number of sides are 24 ft. and 30 ft., respectively. A side of the smaller is 3 ft. What is the side of the other?

2. Prove that if all the diagonals are drawn from any vertex of a regular polygon, they will divide the angle at that vertex into equal parts.

3. Prove that the apothem of a regular polygon bisects any side to which it is drawn.

4. If tangents are drawn at the mid-points of the arcs formed by the sides of a regular inscribed polygon, they form a regular circumscribed polygon of the same number of sides.

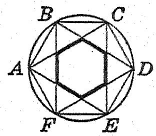


5. In Ex. 4, prove that the sides of the inscribed and circumscribed polygons are parallel.

6. Tangents at the vertices of a regular inscribed polygon form a regular circumscribed polygon.

7. The diagonals AC, BD, CE, etc. of a regular hexagon form another regular hexagon.

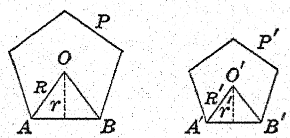
SUGGESTION. — Show that a circle can be inscribed in the smaller hexagon.



***8.** In Ex. 4, the vertices of the circumscribed polygon lie on the radii produced of the inscribed polygon. (Recall postulate 10.)

***9.** If the area of an equilateral triangle is $108\sqrt{3}$ sq. in., find its apothem.

398. Equal ratios in similar regular polygons. If regular polygons P and P' have the same number of sides, we know they are similar. Can you prove that their perimeters have the same ratio as their radii and also as their apothems?

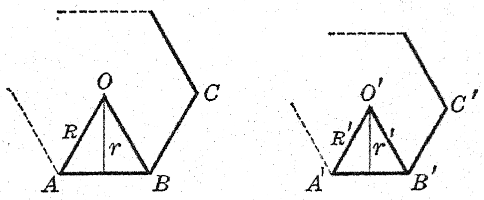


Why is $\triangle OAB \sim \triangle O'A'B'$ (§ 312)?

Then how do the equal ratios follow (§ 306)?

PROPOSITION 7. THEOREM

399. *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*



Given: Regular polygons $ABC\ldots$ and $A'B'C'\ldots$, each of n sides, with centers O and O', perimeters p and p', radii R and R', and apothems r and r'.

To prove: $\dfrac{p}{p'} = \dfrac{R}{R'} = \dfrac{r}{r'}$.

Plan: Prove that $\triangle$ AOB and $A'O'B'$ are similar.

Proof:

STATEMENTS	REASONS
1. Draw radii to A, B, A', B'. $OA = OB$ and $O'A' = O'B'$.	1. *Why?*
2. $\angle AOB = \angle A'O'B'$.	2. § 392.
3. $\dfrac{OA}{O'A'} = \dfrac{OB}{O'B'}$.	3. *Ax.* 5.
4. $\triangle AOB \sim \triangle A'O'B'$.	4. § 312.

Complete the proof.

400. COROLLARY. *The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.*

EXERCISES

1. Find the ratio of the perimeters and the ratio of the areas of two regular hexagons if their sides are 2 in. and 6 in.

2. Squares are inscribed in two circles of radii 2 in. and 8 in., respectively. Find the ratio of the perimeters of the squares and also of their areas.

3. Two regular pentagons have corresponding sides 3 in. and 9 in. What is the ratio of their radii? Of their perimeters? Of their areas?

4. What is the ratio of the perimeters of two regular octagons whose areas are 144 sq. in. and 324 sq. in.?

5. The perimeters of a square and of a regular octagon inscribed in a circle whose radius is 1 in. are 5.66^- in. and 6.12^+ in., respectively. Find to the nearest unit the perimeter of the square and of the regular octagon inscribed in a circle whose radius is 5 in.

6. The area of a regular hexagon inscribed in a circle whose radius is 5 in. is 64.95^+ sq. in. Find to one decimal place the area of a regular hexagon inscribed in a circle whose radius is 1 in.

7. Two regular hexagons have sides $4\frac{1}{3}$ in. and $6\frac{1}{2}$ in. Find the ratio of their perimeters and areas.

---

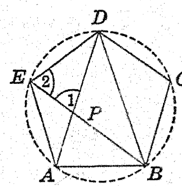
8. What is the ratio of the perimeters of the inscribed and circumscribed equilateral triangles of a circle? (See Ex. 8, § 395.) What is the ratio of their areas?

9. Prove that the area of an inscribed square of a circle is equal to half the area of the circumscribed square.

10. The area of a regular inscribed hexagon is three-fourths the area of a regular circumscribed hexagon of the same circle.

11. The diagonals of a regular pentagon are equal.

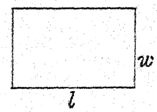
12. If two diagonals of a regular pentagon intersect, the greatest segment of each is equal to a side of the pentagon.



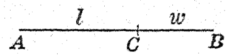
HINT. — Prove $\angle 1 = \angle 2$.

***13.** One side of a regular octagon is 8 in. Find its apothem and area.

401. A. The problem of the golden section. That rectangle is most pleasing to the eye whose base and altitude are so related that their ratio is approximately 5 : 8. Such a rectangle has the ratio of its half perimeter to its length the same as the ratio of its length to its width.



If the segment AB is divided at C so that $\frac{AB}{l} = \frac{l}{w}$, it is said to be divided in **extreme and mean ratio.** This division of a segment greatly interested the Greeks, who thought it the most artistic division of a segment and called it the **golden section.** They also spoke of the proportion as the **divine proportion.**



Two methods of construction were given by Euclid, one thought to be his own, and the one used in this book due to Pythagoras.

EXERCISES A

1. If the base of a rectangle is 10 and its altitude is h, so that $\frac{10+h}{10} = \frac{10}{h}$, find h.

SOLUTION. — (1) $10h + h^2 = 100$. Clearing of fractions.
(2) $h^2 + 10h + 25 = 125$. Completing the square.
(3) $h + 5 = 11.18$, or $h = 6.18$.

2. Is the ratio of the altitude to the base in Ex. 1 nearer to $\frac{5}{8}$ or to $\frac{3}{5}$?

3. If the altitude of a rectangle is 16 in., what must its base be to have the pleasing proportions given in § 401?

4. If AB in § 401 is 10, find l and w.

HINT. — Let $w = 10 - l$.

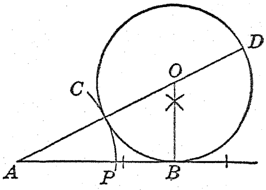
5. A segment a inches long is divided in extreme and mean ratio. Show that the segments are $\frac{a}{2}(3 - \sqrt{5})$ and $\frac{a}{2}(\sqrt{5} - 1)$.

CONSTRUCTION XXV

402. *To divide a segment in extreme and mean ratio.*

Given: Segment AB.

Required: Divide AB in extreme and mean ratio.



Construction: Construct a $\perp$ to AB at B and on it take $BO = \frac{1}{2}\ AB$. Construct a circle with O as a center and BO as radius and let AO intersect the circle at C and D. On AB take $AP = AC$. Then P is the required point of division.

Proof:

STATEMENTS	REASONS
1. AB is tangent to $\odot\ O$.	1. *Why?*
2. $\frac{AD}{AB} = \frac{AB}{AC}$.	2. *Why?*
3. $\frac{AD - AB}{AB} = \frac{AB - AC}{AC}$.	3. *Why?*
4. $CD = AB$ and $AC = AP$.	4. *Why?*
5. Hence $\frac{AP}{AB} = \frac{PB}{AP}$ or, $\frac{AB}{AP} = \frac{AP}{PB}$.	5. *Why?*

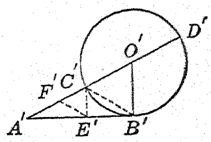
EXERCISES A

1. If the length and width of a picture are obtained by dividing the semiperimeter in extreme and mean ratio, find to the nearest tenth the width of a picture whose length is 25 in.

2. Use the method of § 402 to divide a segment 2 in. long in extreme and mean ratio.

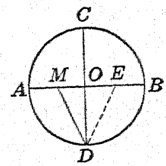
3. Show that the parts of a segment 15 in. long divided in extreme and mean ratio are 9.27 in. and 5.73 in. approximately.

***4.** A segment AB is divided in extreme and mean ratio. Given the longer part AP, construct AB.



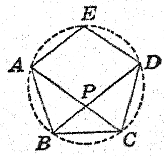
SUGGESTION. — If the figure represents the completed construction, $\angle A'$ can be found by drawing a right triangle whose leg $O'B' = \frac{A'B'}{2}$, where $A'B'$ is any length.

To locate O, notice that $\triangle O'C'B'$ is isosceles. If a $\perp C'E'$ is drawn from C', and $C'F'$ is taken equal to $C'E'$, why will $F'E'$ be $\parallel C'B'$?



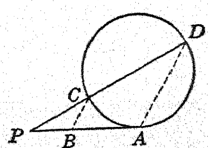
*5. Diameters AB and CD are perpendicular, and M is the mid-point of AO. If $ME = MD$, prove that radius OB is divided in extreme and mean ratio at E.

SUGGESTION. — Show that $OE = \frac{r}{2}(\sqrt{5} - 1)$ where r is the radius. Hence, by Ex. 5, § 401, r is divided in extreme and mean ratio.

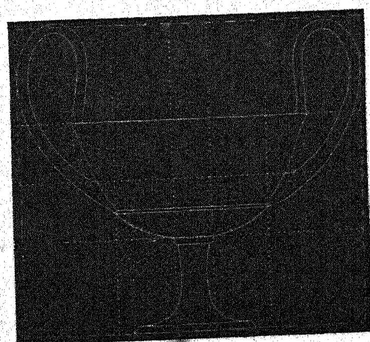


*6. Prove that the diagonals of a regular pentagon divide each other in extreme and mean ratio.

HINT. — Prove that $\triangle ABC \sim \triangle BPC$.



*7. If tangent PA is drawn from any point P to a circle and divided at B in extreme and mean ratio so that BA is the greater segment, and if $PC = BA$, prove that $\overline{CD}^2 = PC \times PD$.



•JAY HAMBIDGE: *The Greek Vase*

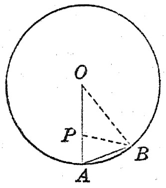
A GRECIAN VASE

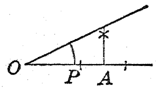
CONSTRUCTION XXVI

403. A. *To inscribe a regular decagon in a circle.*

Given: Circle O.

Required: Inscribe a regular decagon in $\odot\ O$.





Construction: Draw OA and divide it at P so that $OA : OP = OP : AP$. Take $AB = OP$. Then AB is the side of the required decagon.

Proof:

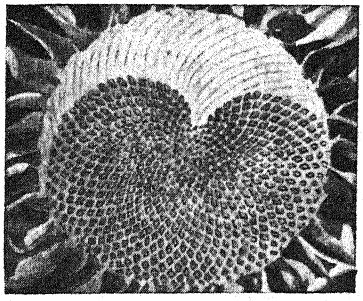
STATEMENTS	REASONS
1. Draw PB and OB. Since $OA : OP = OP : AP$	1. *Construction.*
2. $OA : AB = AB : AP$.	2. *Why?*
3. Hence, since $\angle A = \angle A$, $\triangle OAB \sim \triangle PAB$.	3. § 312.
4. $\therefore \triangle PAB$ is isosceles.	4. *Why?*
5. $\therefore AB = PB = OP$.	5. *Why?*
6. $\angle APB = 2\angle O$.	6. § 128.
7. $\angle A = 2\angle O$.	7. *Why?*
8. $\angle OBA = 2\angle O$.	8. *Why?*
9. $\angle O + \angle A + \angle OBA = 180°$.	9. § 123.
10. $5\angle O = 180°$.	10. *Why?*
11. $\angle O = 36°$.	11. *Ax.* 5.
12. $\angle O$ is one tenth of $360°$.	12. *Why?*
13. $\overset{\frown}{AB}$ is one tenth of the $\odot$.	13. *Why?*
14. $\therefore AB$ is a side of a regular inscribed decagon.	14. *Why?*

404. A. COROLLARY 1. *Regular polygons of 5, 10, 20, 40, 80, etc., sides may be inscribed in a circle.*

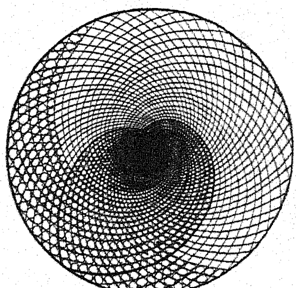
NOTE. — If the alternate vertices of a decagon are joined, a pentagon is formed; if the arcs are bisected and chords drawn, a 20-sided polygon; etc.

THE DIVINE PROPORTION IN PLANTS[1]

The way the seeds are arranged in a sunflower, and the order of arrangement of the leaves on the stem of a

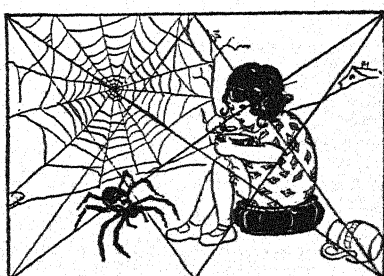


PHOTOGRAPH OF THE SUNFLOWER



GEOMETRIC DRAWING OF SPIRALS IN THE SUNFLOWER

plant, follow a very definite law represented by one of the fractions in the series, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{5}{13}$, $\frac{8}{21}$, $\frac{13}{34}$, etc. It is interesting to notice that these fractions represent successive approximations to the length of the shorter segment of a line 1 in. long divided in extreme and mean ratio; that is, the value of $\frac{1}{2}(3 - \sqrt{5})$ as found in Ex. 5, § 401. Look up the *Phyllotactic law* in the encyclopedia.



This illustration shows how the points of interest in a painting, drawing, or photograph are located by the intersections of the diagonals with perpendiculars drawn from the vertices. Notice some paintings and see if they seem to follow this law.

[1] See *The Art of Composition*, by MICHEL JACOBS.

EXERCISES

(OPTIONAL)

1. If R is the radius of the circle, the side of an inscribed equilateral triangle is $R\sqrt{3}$.

2. If r is the radius of the circle, the side of a circumscribed equilateral triangle is $2\,r\sqrt{3}$.

3. If r is the radius of the circle inscribed in an equilateral triangle, show that the area of the triangle is $3\,r^2\sqrt{3}$.

4. If R is the radius of the circle circumscribed about an equilateral triangle, show that the area of the triangle is $\frac{3\,R^2}{4}\sqrt{3}$.

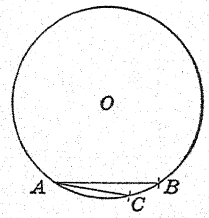
5. If R is the radius of the circle, the area of a regular inscribed hexagon is $\frac{3\,R^2}{2}\sqrt{3}$.

6. The side of a regular circumscribed hexagon is $\frac{2\,r}{3}\sqrt{3}$, when r is the radius of the circle.

***7.** In the figure of § 402, if point Q is taken on BA produced so that $AQ = AD$, prove that segment AB is divided externally in extreme and mean ratio so that $AB : AQ = AQ : QB$.

CONSTRUCTION XXVII

405. A. *To inscribe a regular fifteen-sided polygon in a circle.*



Given: Circle with center O.

Required: Inscribe a regular fifteen-sided polygon in the circle.

Construction: Construct AB, the side of a regular inscribed hexagon, and AC, the side of a regular inscribed decagon. Then CB is the side of the required polygon.

Proof:

SUGGESTION.—What part of the circle is $\overset{\frown}{AB}$? $\overset{\frown}{AC}$? Then what part of the circle is $\overset{\frown}{BC}$? A fifteen-sided polygon is called a **pentadecagon.**

406. A. Corollary. *Regular polygons of 15, 30, 60, 120, etc., sides may be inscribed in a circle.*

EXERCISES A

1. Tell how to construct an angle of 24°; of 12°.

2. Tell how to construct an angle of 6°; of 54°; of 27°.

3. Tell how to construct an angle of 9°; of 3°.

4. Tell how to divide a right angle into five equal parts.

***5.** Find the area of a regular pentagon whose side is 10 in.

***6.** A regular decagon is inscribed in a circle whose radius is 10 in. Find the length of a side of the decagon.

EXERCISES A

(Optional)

For exercises 1–8 use §§ 307–310.

1. If the radius of the circle inscribed in an equilateral triangle is 8 in., find the area of the triangle.

2. If the radius of a circle is 10 in. find the side of an equilateral triangle inscribed in the circle.

3. If the radius of the circle circumscribed about an equilateral triangle is 10 in., find the area of the triangle.

4. If the radius of a circle is 24 ft., find the side of an equilateral triangle circumscribed about the circle.

5. The radius of a circle is 10 in. Find the side of the regular inscribed octagon.

6. Find the apothem of the octagon in Ex. 5.

7. Find the area of the octagon in Ex. 5 and 6.

8. If the radius of a circle is 20 in., find a side of the regular inscribed decagon, and also the side of a regular pentagon inscribed in the same circle.

---

9. Prove that the perpendicular bisector of a side of a regular inscribed pentagon passes through a vertex.

***10.** Construct a regular decagon, having given a side.

***11.** Construct a regular pentagon, having given a side.

***12.** Construct a regular pentadecagon, having given a diagonal connecting two alternate vertices.

407. Construction of regular polygons. An interesting question is: What regular polygons can we construct by means of straightedge and compasses alone? Until the beginning of the last century it was thought that these were limited to the ones we have studied in this course. (§§ 378–9, 384–5, 403–6.)

In 1801 Gauss, a German mathematician, made a study of this question. He found that certain polygons with n sides, where n was a *prime number*, could be constructed if and only if, n was of the form $n = 2^m + 1$ (m is any positive integer). Thus when $m = 1, n = 3$; when $m = 2, n = 5$; when $m = 4, n = 17$; when $m = 8, n = 257$; etc. Hence polygons with 3, 5, 17, etc., sides can be constructed. But when $m = 3, 5, 6, 7$, etc., $n = 9, 33, 65, 129$, etc. These values of n are not *prime numbers*.

Of course, by bisecting the arcs formed by the sides of polygons which can be constructed, we can form other polygons with an even number of sides, and by such combinations as we used in § 405 other polygons can be formed.

Measurement of a Circle

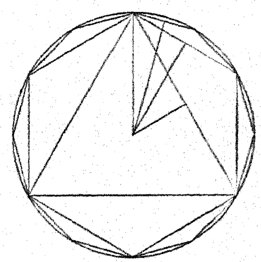
408. Length and area of a circle. The length of a circle cannot be measured as can the length of a segment of a straight line, because a unit of length cannot be laid off along a circle.

For the same reason we cannot apply a square unit of measure to determine the area of the surface inclosed by a circle. However, these measurements can be made as accurately as we please, by the use of certain principles discussed in the following sections.

We shall use the term **circumference** to mean the

length of a circle, and the area of the *surface* inclosed by a circle we shall call its **area**.

409. Increasing the number of sides of regular polygons. We have seen that in two regular polygons with the same number of sides *the perimeters have the same ratio as the radii.* It is evident that the hexagon in the figure has a greater perimeter than the triangle; and that the perimeter of the 12-sided polygon is greater than that of the hexagon. Thus, as the number of sides of the polygon increases, its perimeter gets larger (but is always less than the circumference) and approaches as close as you please to the circumference of the circle.



Also, as the number of sides increases, the apothem grows larger and approaches as close as you please to the radius of the circle.

In a similar manner the area of the inscribed polygon increases as you increase the number of sides, and approaches as close as you please to the area of the circle.

410. As a result of the reasoning above, we may state a principle which can be considered fundamental in finding the circumference and area of a circle.

POSTULATE 19. *Any theorem which has been proved true regarding a regular polygon, and which does not depend on the number of sides of the polygon, is equally true for the circle.*

411. Circumference of a circle. Hence, from § 410 and from § 399, we have:

Theorem: *The circumferences of two circles have the same ratio as their radii.* (Prop. 8.)

412. Corollary 1. *The circumferences of two circles have the same ratio as their diameters.*

Suggestion. — Since by § 411, in circles with circumferences c and c', radii r and r', and diameters d and d':

$$\frac{c}{c'} = \frac{r}{r'}, \text{ why is } \frac{c}{c'} = \frac{d}{d'}?$$

413. Corollary 2. *The ratio of the circumference to the diameter of a circle is a constant: that is, it is the same for any two circles.*

Proof: Since $\frac{c}{c'} = \frac{d}{d'}$ (§ 412).

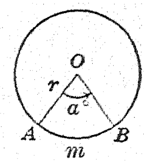
$$\frac{c}{d} = \frac{c'}{d'}. \quad \text{Why?}$$

414. The constant number π. The constant ratio of the circumference to the diameter of a circle is denoted by the Greek letter π (pi). If c is the circumference and d the diameter of a circle, $\frac{c}{d} = \pi$. The value of π is 3.1416^-.

415. Formula for the circumference. Since $\frac{c}{d} = \pi$ for any circle, then $c = \pi d$, and since $d = 2\,r$, we have $c = 2\,\pi r$. Hence the theorem:

Corollary 3. *The circumference of a circle is equal to the product of its radius by twice the constant number π or, $c = 2\,\pi r$.*

416. Corollary 4. *The length of an arc of a circle in linear units has the same ratio to the circumference, as the number of degrees in the arc has to 360.*



Given: Circle O, with circumference c, radius r, and $\widehat{AB}$ containing m linear units and intercepted by $\angle O$ containing a degrees.

To prove: $\dfrac{m}{2\pi r} = \dfrac{a}{360}$.

Proof: Since two arcs have the same ratio as their central angles, and $\widehat{AB}$ contains a arc degrees, while the circumference c contains 360, $\dfrac{\widehat{AB}}{c} = \dfrac{a}{360}$. But $\widehat{AB} = m$ linear units and $c = 2\pi r$. Hence, $\dfrac{m}{2\pi r} = \dfrac{a}{360}$.

EXERCISES

Use $\pi = 3.14$.

1. Find the circumference of a circle whose radius is 5 ft.

2. If the circumference of a circle is 24 ft., find its diameter.

3. Find the radius of a circle whose circumference is equal to the semicircumference of a circle whose radius is 4 in.

4. The circumference of a circle is 147 in. Find the length of an arc of 120°.

5. In a circle whose radius is 8 ft. find the length of an arc whose central angle is 36°.

6. Archimedes proved that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. Find to how many decimal places these numbers approximate the value of π (3.14159+).

7. A water tank is cylindrical in form and 20 ft. in diameter. How long a piece of strap iron is required to make a band around it, allowing 1 ft. for overlapping?

8. How is the circumference of a circle affected if its radius is multiplied by 2? By 5? By $\frac{1}{7}$?

9. The length of an arc of a circle is 33 in. If the arc is 54°, what is the radius of the circle?

10. If the radius of a circle is 25 in., and an arc is 60 in., how many degrees does the arc contain?

11. The central angle whose arc is equal to the radius is used in mathematics as a unit of measure of angles, and is called a *radian.* Find the number of degrees in a radian.

Area of a Circle

417. From the reasoning in §§ 409 and 410, and because the *area of a regular polygon is half the product of its perimeter by its apothem,* we have:

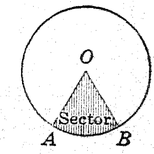
Theorem: *The area of a circle is equal to half the product of its circumference by its radius.* (Prop. 9.)

418. Corollary 1. *The area of a circle is equal to the product of the constant number* π *by the square of the radius.*

Proof: Since

$$c = 2\,\pi r, \text{ and } A = \frac{cr}{2}, \text{ then } A = \frac{2\,\pi r \times r}{2} = \pi r^2.$$

419. A **sector** of a circle is that part of the interior of a circle bounded by two radii and an arc. $\angle AOB$ is called the angle of the sector.

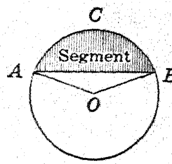


420. Corollary 2. *The area of a sector has the same ratio to the area of the circle as the angle of the sector has to* 360°.

421. Corollary 3. *The area of a sector is equal to the product of the length of its arc by half the radius.*

422. Corollary 4. *The areas of two circles have the same ratio as the squares of their radii, or as the squares of their diameters.*

423. A **segment of a circle** is that part of the interior of a circle bounded by a chord and its arc.



The area of a segment can be found by subtracting from the area of the sector $OACB$ the area of $\triangle OAB$.

EXERCISES

Use $\pi = 3.14$.

1. The radius of a circle is 4 in. Find its area.

2. If the radius of a circle is doubled, how is its area affected?

3. The area of a circle is 254.34 sq. in. Find its diameter.

4. Find the area of a circular sector if its arc is 30° and the radius of its circle is 15 in.

5. An arc of a sector is 18 in. long. Find the area of the sector if the radius of the circle is 12 in.

6. How many degrees in the angle of the sector in Ex. 5?

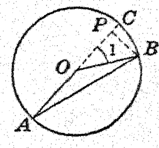
7. A cow is tied at the end of a rope 100 ft. long which is fastened to the corner of a barn 50 ft. square. Over how great an area can the cow graze?

---

8. If the diameter of a circle is d, prove that its area is $\frac{1}{4}\pi d^2$.

9. In the papyrus that he copied, Ahmes, the Egyptian, said the area of a circle could be found by squaring $\frac{8}{9}$ of the diameter. What value of π is indicated in this rule?

10. Show that a circle has the same area as a triangle whose base equals the circumference, and whose altitude equals the radius.



11. Find the area of a segment of a circle whose arc is 150° if the radius of the circle is 30 in.

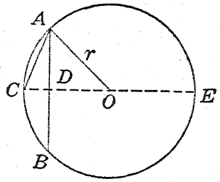
Hint. — If $BP \perp AO$ produced, how large is $\angle 1$? Then how long is PB?

12. The arc of a segment of a circle is 120°. Find the area of the segment if the radius is 15 in. (See § 160.)

13. In a circle whose radius is 5 in. a sector has an area of 25 sq. in. Find the area of a similar sector in a circle whose radius is 8 in.

Note. — Similar sectors are sectors having equal central angles.

424. A. Problem. *Given the radius of a circle and the side of a regular inscribed polygon, find the side of a regular inscribed polygon with twice as many sides.*



Given: Circle O, with radius r, and AB the side of a regular inscribed polygon.

To find: The side of a regular inscribed polygon with twice as many sides.

SUGGESTIONS. — Let $AB = s$, and let AC equal the side of a regular polygon of twice as many sides. Draw diameter CE intersecting AB at D.

Prove that CE is the perpendicular bisector of AB.

In $\triangle ADO$, show that $\overline{DO}^2 = r^2 - \frac{1}{4}s^2$.

Hence $DO = \sqrt{r^2 - \frac{1}{4}s^2}$, and $CD = r - \sqrt{r^2 - \frac{1}{4}s^2}$. Why?

Now by § 324 (b) $\overline{AC}^2 = CE \times CD = 2r(r - \sqrt{r^2 - \frac{1}{4}s^2})$. Explain.

Hence, $AC = \sqrt{2r(r - \sqrt{r^2 - \frac{1}{4}s^2})} = \sqrt{2r^2 - r\sqrt{4r^2 - s^2}}$.

Write the proof in full.

425. A. Computation of π. Using the formula in § 424 the approximate value of π may now be computed.

For this purpose, in a circle of radius $r = 1$, let regular polygons of 6, 12, 24, 48, etc., sides be inscribed.

If s_6 is the side and p_6 the perimeter of the polygon of 6 sides, and s_{12} is the side and p_{12} the perimeter of the polygon of 12 sides, etc., then $s_6 = 1$ and $p_6 = 6$. Why?

Then by substituting successively in the formula $s_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - s_n^2}}$, where s_{2n} is the side of a polygon of twice as many sides as s_n, we find that the perimeter of a polygon of 768 sides is 6.283169+. If this value is taken as approximately equal to the circumference, then dividing it by the length of the diameter, 2, gives 3.14158+. A closer approximation to π, obtained by taking a polygon with a greater number of sides is 3.14159.

The values of π used generally in computation are 3.1416, 3.142, 3.14, or $\frac{22}{7}$, depending upon the accuracy of the result required.

NOTE. — The exact value of π can be computed correct to as many places as we wish. Its value to 25 decimal places is 3.14159265358979323846 26433. The value has been computed to 707 decimal places.

The ratio π has a long and interesting history. The ancient Babylonians and Hebrews thought that $\pi = 3$, as shown by the Bible. (See 1 Kings vii, 23, and 2 Chronicles iv, 2.) Ahmes, the Egyptian, about 1700 B.C., used a value of π equivalent in modern notation to 3.1604. Archimedes, a Greek mathematician (225 B.C.), by employing inscribed and circumscribed regular polygons, proved that the value of π is between $3\frac{1}{7}$ and $3\frac{10}{71}$. Aryabhatta, a Hindu born 476 A.D., showed that $\pi = 3.1416$. Other approximations for π have been used by different people in the past.

The attempt to compute the exact value of π was connected with the famous impossible problem with which mathematicians struggled for centuries, dating from the time of the ancient Greeks, namely, to construct with straightedge and compasses alone a square whose area should equal that of a given circle, or to "square a circle."

426. Summary of the Work of Unit Eight.

In this unit you have learned about:

I. ***Regular polygons.***

1. *If a circle is divided into equal arcs the chords of the arcs form a regular inscribed polygon and the tangents at the points of division form a regular circumscribed polygon.*

2. *A circle can be circumscribed about or inscribed in any regular polygon.*
3. *Regular polygons of the same number of sides are similar; their perimeters have the same ratio as their radii or as their apothems; their areas have the same ratio as the squares of their radii or as the squares of their apothems.*
4. *Regular polygons of 3, 6, 12, 24, ..., of 4, 8, 16, 32, ..., of 5, 10, 20, 40, ..., of 15, 30, 60, 120, ..., sides can be inscribed in a circle.*

II. ***Constructions.***

1. *To inscribe a square in a given circle.*
2. *To inscribe a regular hexagon in a given circle.*
3. *To divide a segment in extreme and mean ratio.*
4. A. *To inscribe a regular decagon in a circle.*
5. A. *To inscribe a regular fifteen-sided polygon in a circle.*

III. ***Measurement of a circle.***

1. *Circumference of a circle:* $C = 2\pi r$ or πd.
2. *Length of arc of a circle:* $L = \frac{\text{angle of arc}}{360} \times 2\pi r$.
3. *Area of a circle:* $A = \pi r^2$ or $\frac{1}{4}\pi d^2$.
4. *Area of a sector:* $A = \frac{\text{angle of sector}}{360} \times \pi r^2$.

REVIEW OF UNIT EIGHT

See if you can answer the questions in the following exercises. If you are in doubt look up the section to which reference is made. Then study that section before taking the test. The references given are those most closely related to the exercise.

1. What is a regular polygon? § 377.

2. Are two regular polygons always similar? § 397.

3. Is an inscribed equilateral polygon always a regular polygon? § 382.

4. What is the center of a regular polygon? § 392.

5. Define apothem and radius of a regular polygon. § 392.

6. How large is each central angle of a regular polygon of 5 sides? Of 6 sides? Of n sides? § 392.

7. If a regular polygon has n sides, is one of its angles equal to $\frac{n-2}{n} \times 180°$? § 393.

8. How can you find the number of sides of a regular polygon if you are given a central angle? An exterior angle? An interior angle? §§ 392, 393.

9. What are the names of regular polygons of 3, 4, 5, 6, and 8 sides, respectively? § 131.

10. Two regular polygons have the same number of sides. If a side of one is S and of the other is s, what is the ratio of their perimeters? Of their radii? Of their apothems? Of their areas? §§ 399, 400.

11. The side of a regular inscribed hexagon is r, and the side of a regular circumscribed hexagon is $\frac{2}{3} r\sqrt{3}$. What is the ratio of their areas? § 400.

12. In Ex. 11 what is the ratio of the apothems? § 400.

13. Two circles have diameters D and d. What is the ratio of their radii? Of their circumferences? Of their areas? §§ 412, 422.

14. How many pipes $\frac{1}{2}$ in. in diameter are needed to replace a 1-in. pipe? § 418.

15. Is the side of an inscribed equilateral triangle half the side of the circumscribed triangle? § 395, Ex. 8.

16. Two circles are concentric. If the radius of the larger is R, and of the smaller r, what is the area of the circular ring? § 418.

17. How can you find the area of a regular polygon? § 395.

18. How can you find the area of a circle if you know the circumference? § 418.

19. How can you find the circumference of a circle if you know the area? § 418.

20. Give two methods for finding the area of a sector. §§ 420, 421.

21. How can the area of a segment of a circle be found? § 423.

22. Can the area of a segment of a circle always be found? § 423.

23. One circle has an area nine times that of another circle. How do their radii compare? § 422.

24. Can you find the length of an arc of a circle if you are given its central angle and its radius? § 416.

25. If, in two unequal circles, equal central angles are taken, what is the ratio of the intercepted arcs? §§ 411, 416.

26. If, in two equal circles, unequal central angles are taken, what is the ratio of their intercepted arcs? § 416.

27. In a circle a central angle O intercepts a certain chord AB. Will a central angle twice as large as $\angle O$ have a chord twice as large as AB?

28. Will 100 feet of fence inclose a greater area in the form of an equilateral triangle, a square, or a circle? §§ 341, 363, Ex. 5, 418.

29. As you increase the number of sides of a regular inscribed polygon, how does the apothem change? The perimeter? The area? § 409.

30. As you increase the number of sides of a regular circumscribed polygon, tell how each of the parts of the polygon changes. § 409.

31. What regular polygons can you inscribe in a circle? § 407.

32. How can you inscribe in a circle a regular polygon of eight sides? Of twelve sides? §§ 378, 379, 383.

***33.** Is the ratio of the areas of two similar sectors (sectors having equal central angles) the same as the ratio of the squares of their radii?

***34.** Is the ratio of the areas of two similar sectors the same as the ratio of the squares of the lengths of their arcs?

NUMERICAL EXERCISES

1. Find the area of a sector of a circle with central angle 40° and radius 12 in.

2. How long is the tire of a carriage wheel that is 4 ft. in diameter?

3. How many revolutions per mile does a 28-in. bicycle wheel make?

4. The earth is nearly 8000 miles in diameter. What is the approximate length of the equator?

5. A circular running track is one mile long. If two athletes run, one 1 ft. from the pole (inner curb) and the other 6 ft. from it, how much farther does the second man run?

6. How much farther does the second man in Ex. 5 run if the running track consists of two parallel straightaways each a quarter of a mile long, and two semicircular ends each a quarter of a mile long at the inner curb?

7. The boiler of an engine has 96 flues, each 3 in. in diameter, which conduct the hot air from the furnace through the water to heat it. If the rule requires that the smokestack shall have the same capacity (cross-sectional area) as all of the flues which empty the smoke into it, what must be the diameter of the smokestack?

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8. Find the area of the regular hexagon inscribed in a circle whose radius is 36 in.

9. Find the area of a square inscribed in a circle with radius 12 in.

10. Find the area of the square circumscribed about the circle in Ex. 9.

11. Find the area of the regular hexagon circumscribed about the circle in Ex. 8.

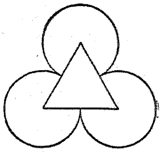
12. An isosceles triangle has its vertex at the center of a circle and two of its sides are radii. If the radius of the circle is 5 in., what is the area of the triangle:

a. If the central angle is 150°?
b. If the central angle is 120°?
c. If the central angle is 90°?
d. If the central angle is 60°?
e. If the central angle is 135°?
f. If the central angle is 45°? (See Ex. 11, § 423.)

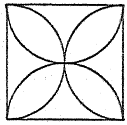
13. Find the areas of the segments formed in Ex. 12.

14. Find the area of a segment of a circle whose arc is 60°, the radius being 3 ft.

15. A horizontal oil tank 8 ft. in diameter is filled with oil to a depth of 2 ft. Find the area of a segment formed by a vertical cross section of the oil.



16. A **trefoil** is constructed, as in the figure, by drawing arcs of circles with centers at the vertices of an equilateral triangle and radii equal to one half of the side of the triangle. If the side of the triangle is 12 in., find the area of the trefoil, that is, of the entire surface inclosed by the three arcs. Find its perimeter.



17. The **quatrefoil** shown in the figure is formed by the arcs of four semicircles drawn with the sides of a square as diameters. Find the area and perimeter of the quatrefoil if the side of the square is 4 in.

18. The perimeter of a church window is formed by three equal semicircles drawn on the sides of an equilateral triangle as diameters. If the sides of the triangle are 4 ft. long, find the area of the window and the length of its perimeter.

19. Consider the earth a perfect sphere, and a circular hoop made whose circumference is 6 ft. longer than the equator of the earth. If this hoop were placed around the equator with its center at the center of the earth, what would be the width of the ring between them?

20. With each vertex of an equilateral triangle as a center and with a radius equal to half of a side, an arc is drawn within the triangle, terminating in the sides. If the side of the triangle is 6 in., find to two decimal places the area and perimeter of the surface inclosed by the three arcs.

CONSTRUCTIONS

Tell how to inscribe the following regular polygons in a circle:

1. Equilateral triangle

2. Square

3. Hexagon

4. Octagon

Tell how to construct:

5. A straight line tangent to a given circle at a given point on the circle.

6. A straight line tangent to a given circle and passing through a given external point.

Tell how to circumscribe the following regular polygons about a circle:

7. Equilateral triangle

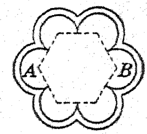
8. Square

9. Hexagon

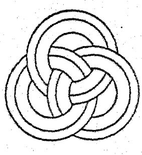
10. Octagon

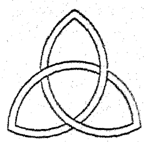
11. The top of a taboret is to be made in the form of a regular octagon whose longest diagonal is 12 in. Construct a design of it to the scale of 3 in. to an inch.

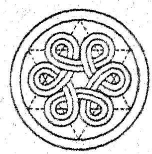
12. Construct a paper pattern for a doily like the adjoining figure, making the width from A to B 12 in.



13–15. Study the following ornamental designs and explain how they are made. Construct designs, similar to these, making the drawings several times as large.







16–18. Copy the arches shown below. Fig. 1 is a Persian arch, Fig. 2 a Gothic window, and Fig. 3 a lancet arch.

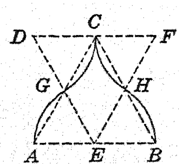


FIG. 1

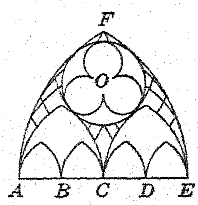


FIG. 2

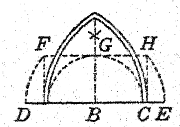
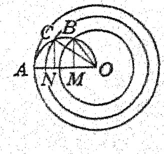


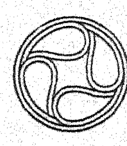
FIG. 3

19. Divide a given circle into three equal parts by drawing concentric circles.

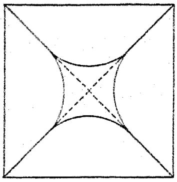
SUGGESTION. — Trisect the radius at M and N. Draw a semicircle on OA as diameter. Erect perpendiculars MB and NC, meeting the semicircle at B and C. With centers O and radii OB and OC, draw circles.

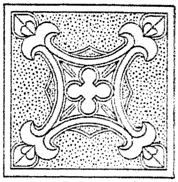


20. The *rosette* shown is formed by arcs of tangent and intersecting circles. See if you can make a similar rosette. Make the diameter of the largest circle 3 in.



21. The adjoining figure is of a steel *ceiling panel*. The four tangent arcs are drawn with centers on the sides of the square. Construct a similar design three inches square.



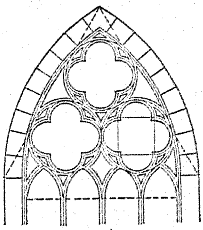


Analysis. — Since the arcs are to be tangent to each other, they must be tangent to what lines? Since each arc is to be tangent to both diagonals, its center must be on what line?

22. Construct within a given square four equal circles each tangent to two others and to two sides of the square.

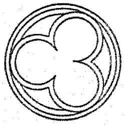
23. In a given circle construct three equal circles, each tangent to the other two and to the given circle.

24. This **tracery** window contains three equal circles within an equilateral triangle, each circle being tangent to the other two circles and to the two sides of the triangle. Construct three such circles within a given triangle.



Suggestion. — Since the circles are tangent to each other, they must have a common tangent line at each point of contact.

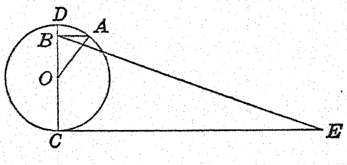
25. The construction of a trefoil, formed by the arcs of three equal circles in a given circle as shown in the figure, is encountered frequently in architectural designs. Explain the construction, and draw a trefoil in a given circle.



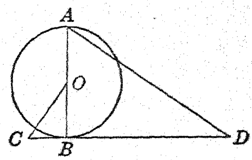
Suggestion. — Use a circle at least 3 in. in diameter. Proceed as in Ex. 23.

***26.** A line segment approximately equal to the circumference of a given circle may be constructed as follows:

Draw diameter CD. Construct $\angle AOD = 30°$. Draw $AB \perp CD$. Draw CE tangent at C and equal to $3\,CD$. Draw BE. Then $BE =$ the circumference, approximately. Determine the accuracy of this construction.



***27.** Mathematicians long attempted to construct a line segment equal in length to the circumference of a given circle. The following is an approximate construction:



Draw the diameter AB, and draw CD tangent at B, making $\angle COB = 30°$, and CD three times the radius. Draw AD. Then $2\ AD$ is the circumference, nearly. Determine the accuracy of this construction by computing the ratio of $2\ AD$ to AB.

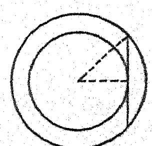
SUGGESTION. — Let R = radius. Compute AB and $2\ AD$ in terms of R, then divide.

GENERAL EXERCISES

1. Show that the area of a circumscribed equilateral triangle is four times the area of the inscribed equilateral triangle.

2. What is the radius of a circle whose area is equal to the sum of the areas of two circles having radii R and r, respectively?

3. Prove that the area of the ring between two concentric circles whose radii are R and r, respectively, is $\pi(R + r)(R - r)$.



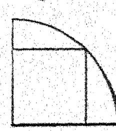
4. Prove that the area of the ring between two concentric circles is equal to the area of a circle whose diameter is a chord of the larger circle which is tangent to the smaller.

SUGGESTION. — In the formula of Ex. 3 the factors $(R + r)\ (R - r)$ equal the square of what single line segment?

5. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of a regular circumscribed polygon of the same number of sides.

HINT. — Use the figure for Ex. 4, § 397. What triangles are similar?

6. Prove that if the radius of a circle is R, the area of a segment of the circle whose arc is a quadrant is equal to $\frac{1}{4} R^2(\pi - 2)$.

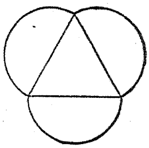


7. The area of the square inscribed in a sector of a circle whose central angle is a right angle is equal to half the area of a square whose side is a radius.

8. What is the ratio of the area of a square inscribed in a semicircle to the area of a square inscribed in the whole circle?

9. Prove that if the radius of a circle is R, the area of a segment of the circle whose arc is $60°$ is equal to $\frac{1}{12} R^2(2\pi - 3\sqrt{3})$.

10. With the mid-points of the sides of an equilateral triangle as centers, and a radius of half the side, arcs are drawn as shown. Find the area of the figure in terms of s, the side of the equilateral triangle.



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11. If all the diagonals joining the alternate vertices of a regular hexagon are drawn, the area of the second regular hexagon which they form is one third that of the original hexagon. (See figure Ex. 7, § 397.)

12. The area of a regular inscribed polygon of twelve sides equals three times the square of the radius. (Use the figure in § 424. Notice that $\angle AOC$ will be $30°$. Then how long is altitude AD?)

13. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

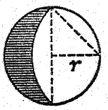
14. If from any point within a regular polygon of n sides perpendiculars are drawn to all of the sides, the sum of these perpendiculars is equal to n times the apothem of the polygon.

15. A. If r is the radius of a regular octagon, prove that the side is $r\sqrt{2-\sqrt{2}}$ and the apothem is $\frac{1}{2}r\sqrt{2+\sqrt{2}}$. (See § 424.)

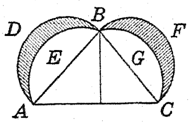
16. A. If r is the radius of a regular polygon of twelve sides, prove that a side is $r\sqrt{2-\sqrt{3}}$ and the apothem is $\frac{1}{2}r\sqrt{2+\sqrt{3}}$. (See § 424.)

***17.** A. The side of a regular circumscribed octagon is less than the radius of the circle, but is greater than three-fourths of the radius. (Use the indirect method.)

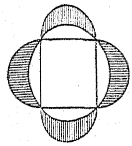
***18.** A crescent is bounded by a semicircle and the arc of another circle whose center is on the first arc produced. If r is the radius of the semicircle, show that the area of the crescent is r^2.



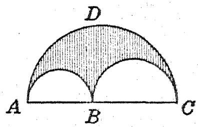
***19.** Semicircles are constructed on the three sides a, b, and c, of a right triangle, as in Fig. I. Show that the sum of the crescents $ADBE$ and $BFCG$ is equal to the triangle.



I



II

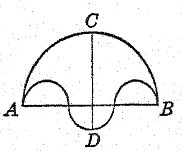


III

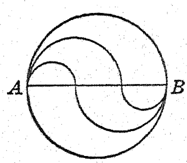
***20.** Hippocrates, a Greek mathematician who lived about 460 B.C., proved that, if on the sides of an inscribed square as diameters semicircles are described, the area of the four crescents lying without the circle equals the area of the inscribed square (Fig. II). Can you prove it?

***21.** The following is an old principle, due to Archimedes: AC (Fig. III) is divided into two unequal parts at B. Semicircles are drawn with AC, AB, and BC, respectively, as diameters. BD is perpendicular to AC and meets the larger semicircle at D. Then the area of the surface bounded by the three arcs AC, AB, and BC is equal to the area of the circle with diameter BD. Prove it.

NOTE. — The curve formed is called the *shoemaker's knife*.



IV



V

***22.** AB is trisected and semicircles are drawn as shown in Fig. IV. Prove that the area of the figure bounded by the curved lines is equal to the area of the circle whose diameter is CD.

NOTE. — The curve formed is called the *salt cellar*.

***23.** Diameter AB (Fig. V) is trisected and semicircles are drawn as shown. Prove that the area of the circle is trisected.

PRACTICAL APPLICATIONS

(OPTIONAL)

1. A grindstone of Ohio stone will stand a surface speed of 2500 ft. per minute. How many revolutions per minute will a stone stand if it is 4 ft. in diameter?

2. An emery stone will safely stand a surface speed of 5500 ft. per minute. An emery grinder is to make 1500 revolutions per minute. What is the largest wheel that may safely be used?

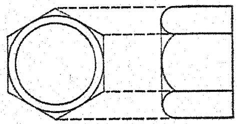
3. The driving pulley of an engine is 6 ft. in diameter and makes 120 revolutions per minute. It is belted to a 24-inch pulley on the main shaft that runs the machinery of a mill. Find the speed of the shaft (revolutions per minute).

4. If it is customary in iron turning to allow a cutting speed at the rim of 40 ft. per minute, at what speed should a lathe be driven for turning a piece of iron 2 in. in diameter?

5. The steam pressure of an engine is indicated as 96 lb. per square inch. The cylinder of the engine is 20 in. in diameter. What is the total pressure against the piston?

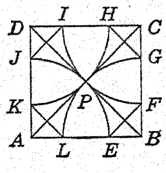
6. What is the propelling pressure exerted against the piston of a locomotive when the steam pressure is 100 lb. per square inch and the diameter of the cylinder is 28 in.?

7. In making a water wheel, a square block of wood is to be made into the form of a regular octagon by cutting off the four corners and then attaching buckets to each of the eight faces. Show how to cut off the corners accurately.

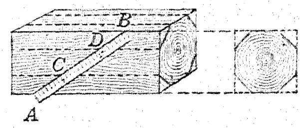


8. Draw the face and edge views of a hexagonal nut of a bolt, the side of the nut to be one inch.

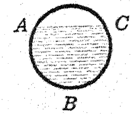
9. A method of obtaining a regular octagon from a square is: Draw the diagonals of the square, intersecting at P. With radius equal to AP and centers A, B, C, D, draw arcs cutting the sides of the square at E, F, G, H, I, J, K, L. Draw EF, GH, IJ, and KL. Then $EFGHIJKL$ is a regular octagon. Prove it.



10. Carpenters, in order to cut a square piece of timber down to an octagonal shape, proceed as follows: Place a 24-inch rule diagonally across the timber, the ends even with the edges at A and B, as in the figure. Then mark the points C and D at the 7-inch and 17-inch points on the rule. Through these points draw lines parallel to the edges of the timber. Repeat this on each face. The corners must be cut down to these lines to form an octagonal piece of timber. Is this method accurate or approximate? Prove the answer.

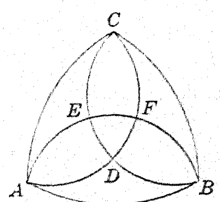


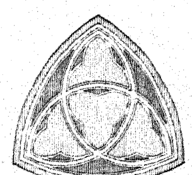
11. A conduit for carrying water is 12 ft. in diameter. If the water is 9 ft. deep in the conduit, find the area of the cross section of the water. This must be known before the rate of flow of the water can be determined.



12. Find the length of the arc ABC in the conduit of Ex. 11. This "wetted perimeter" must be known before the resistance of the conduit to the water flow can be determined.

13. The adjoining figure is much used in different decorative designs, such as ornamental church windows. Arcs AB, AC, and BC are drawn with centers at C, B, and A, the vertices of an equilateral triangle. Arcs $ADFC$, $BDEC$, and $AEFB$ are semicircles.

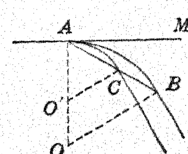




Where are the centers of these semicircles?

14. How much belting does it require to run over two pulleys each 24 in. in diameter and with their centers 16 ft. apart?

15. If the curves AC and AB of two railroad tracks begin at the same tangent point A and end in parallel tracks at C and B, prove that (1) their chords AC and AB coincide in direction, and (2) their chords are proportional to their radii.



PRACTICE TESTS

These are practice tests. See if you can do all the exercises correctly without referring to the text. If you miss any question look up the reference and be sure you understand it before taking other tests.

TESTS ON UNIT EIGHT

TEST ONE

Numerical Exercises

Leave answers in terms of π or use $\pi = 3.14$ as your teacher directs.

1. Find the apothem of a square whose side is 6 ft. § 392.

2. The circumference of a circle is 20π. Find its radius. § 415.

3. The radius of a circle is 5 in. Find its area. § 418.

4. Find to two decimals the radius of a square whose side is 5 in. §§ 326, 392.

5. In a circle whose radius is 3 in., find the area of a sector whose angle is 70°. § 420.

6. The radius of a circle is 5 in. How long is the perimeter of the regular inscribed hexagon? § 379.

7. How many degrees in each central angle of a regular pentagon? § 392.

8. Find the length of the arc of a circle which contains 25 arc degrees, if the radius of the circle is 18 in. § 416.

9. The perimeters of two regular polygons having the same number of sides are 20 in. and 32 in., respectively. A side of the smaller is 2 in. What is a side of the other? § 397.

10. Find the apothem of an equilateral triangle if the altitude is 18 in. §§ 278, 392.

11. Regular pentagons are inscribed in two circles of radii 2 in., and 5 in., respectively. What is the ratio of their areas? § 400.

12. Find the area of a circular segment whose central angle is 90°, if the radius of the circle is 12 in. § 423.

TEST TWO

Matching Exercises

In group A brief descriptions of the terms in group B are given. Match them correctly.

A	B
I. A polygon with five sides. § 131.	1. Pi (π).
II. A polygon whose sides are tangent to a circle. § 219.	2. Apothem of an equilateral triangle.
III. Part of the area of a circle bounded by a chord and its arc. § 423.	3. Hexagon.
IV. Part of the area of a circle cut off by two radii. § 419.	4. Pentagon.
V. A polygon whose sides are chords of a circle. § 198.	5. Regular polygon.
VI. A polygon with equal sides and equal angles. § 377.	6. Apothem.
VII. The length of a circle. § 408.	7. Radius of an equilateral triangle.
VIII. The ratio of the circumference of a circle to its diameter. § 414.	8. Segment of a circle.
IX. The radius of the inscribed circle of a regular polygon. § 392.	9. Inscribed polygon.
X. Two-thirds of the altitude of an equilateral triangle. § 394, Ex. 2.	10. Sector of a circle.
XI. Polygon with six sides. § 131.	11. Circumscribed polygon.
XII. One-third of the altitude of an equilateral triangle. § 394, Ex. 2.	12. Circumference.

TEST THREE

True-False Statements

If a statement is always true, mark it so. If it is not always true, replace each word in italics by a word which will make it a true statement.

1. If the radius of a circle is doubled, the area is multiplied by *two.* § 422.

2. A *segment* of a circle is that part of the area within a circle bounded by a chord and its arc. § 423.

3. The *areas* of two similar regular polygons have the same ratio as any two corresponding sides. § 373.

4. The area of a circle is equal to $\frac{1}{2}\,\pi rc$. § 417.

5. If two regular polygons have the same number of sides they are *congruent.* § 397.

6. If the radius of a circle is doubled, the *circumference* is also doubled. § 411.

7. In an inscribed regular hexagon *two* of the diagonals are diameters. § 379.

8. The apothem of an equilateral triangle is one-third the altitude and the radius is *two-thirds* the altitude. §§ 278, 392.

9. Each angle of a regular polygon of n sides is $n - 2$ straight angles. § 393.

10. Each *central* angle of a regular polygon of n sides contains $\frac{360}{n}$ degrees. § 392.

11. The central angle of a regular polygon is the *complement* of an angle of the polygon. §§ 392, 393.

12. The area of a sector has the same ratio to the area of the circle as the angle of the sector has to *180°*. § 420.

CUMULATIVE TESTS ON ALL EIGHT UNITS

TEST FOUR

Numerical Exercises

1. How many degrees in each interior angle of a regular octagon?

2. The bases of a trapezoid are 8 in. and 12 in; how long is the line joining the mid-points of the non-parallel sides? § 157.

3. What must be the altitude of the trapezoid given in Ex. 2 to make the area 40 sq. in.? § 353.

4. In right triangle ABC, the hypotenuse AB is 10 in.. If AC is 5 in., how large is angle B? § 161.

5. A quadrilateral is inscribed in a circle. If two consecutive angles are 140° and 80°, respectively, find the number of degrees in the other two angles. Page 227, Ex. 14.

6. A tangent and a secant drawn from an exterior point P to a circle form an angle of 25° at P. If the smaller of the two arcs intercepted is 10°, how large is the other arc? § 248.

7. Find the length of the shortest chord that can be drawn through a point 5 in. from the center of a circle whose radius is 13 in. § 210, Ex. 10.

8. The bases of two similar triangles are 6 in and 10 in. The altitude of the first is 8 in. Find the altitude of the second. § 306.

9. The side of an equilateral triangle is 6 in. Find its area. § 376–I.

10. The diagonal of a square is 10 in. Find its area. § 376–I.

11. The sides of two similar triangles are to each other as 5 is to 3. The area of the larger is 50 sq. in. Find the area of the smaller. § 372.

12. A tangent PA and a secant PBC are drawn from a point P to a circle. If PB, the external segment of the secant is 8 in. and chord BC is 10 in., find PA. § 335.

TEST FIVE

True-False Statements

If a statement is always true mark it so. If it is not always true, replace each word in italics by a word which will make it a true statement.

1. The areas of *two* triangles are to each other as the squares of any two corresponding sides. § 372.

2. The area of a trapezoid is equal to the product of the altitude by *half* the sum of its bases. § 353.

3. A median of a triangle divides the triangle into two *equal* triangles. § 350.

4. The area of a triangle equals $s\sqrt{(s-a)(s-b)(s-c)}$. § 358.

5. If a line is drawn through the vertex of an isosceles triangle parallel to the base, it bisects the *exterior* angle at the vertex. §§ 113, 114.

6. In an inscribed square the diagonals are *diameters*. § 378.

7. Parallel lines intercept equal arcs on a *circle*. § 226.

8. If a quadrilateral has its opposite sides *parallel*, its opposite angles are equal. § 141.

9. The diagonals of a rhombus are *equal*. § 162, Ex. 6.

10. The angles at the extremities of the *shortest* side of a triangle are acute. § 170.

11. A line parallel to one side of a triangle cutting the other two sides, forms with the sides a triangle *similar* to the given triangle. § 305.

12. The diagonals of a *trapezoid* divide each other in the same ratio. § 305.

TEST SIX

Matching Formulas

In group B statements are given which are expressed by the formulas in group A. Match them correctly.

A

I. $c^2 = a^2 + b^2$. § 326.	VII. $A = \frac{\text{angle}}{360} \times \pi r^2$. § 420.
II. $A = \frac{1}{2} r \times \text{arc}$. § 421.	VIII. $A = \frac{a^2}{4}\sqrt{3}$. § 376–I.
III. $\frac{p}{p'} = \frac{a}{a'}$. § 399.	IX. $h = \frac{a}{2}\sqrt{3}$. § 376–I.
IV. $A = \frac{h}{2}(b_1 + b_2)$. § 353.	X. $L = \frac{\text{arc}}{360} \times 2\pi r$. § 416.
V. $c = 2\pi r$. § 415.	XI. $d = a\sqrt{2}$. § 376–I.
VI. $\frac{A}{A'} = \frac{a^2}{a'^2}$. § 373.	XII. $a^2 = b^2 + c^2 - 2\,cp^b$. § 328.

B

1. The altitude of an equilateral triangle is half the side times the square root of three.

2. The diagonal of a square is equal to the side times the square root of two.

3. The area of a trapezoid is half the altitude times the sum of the bases.

4. The area of a sector of a circle is equal to $\frac{1}{360}$ times the area of the circle multiplied by the number of degrees in the angle of the sector.

5. The areas of two similar polygons have the same ratio as the squares of two corresponding sides.

6. The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.

7. In any triangle the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other side on it.

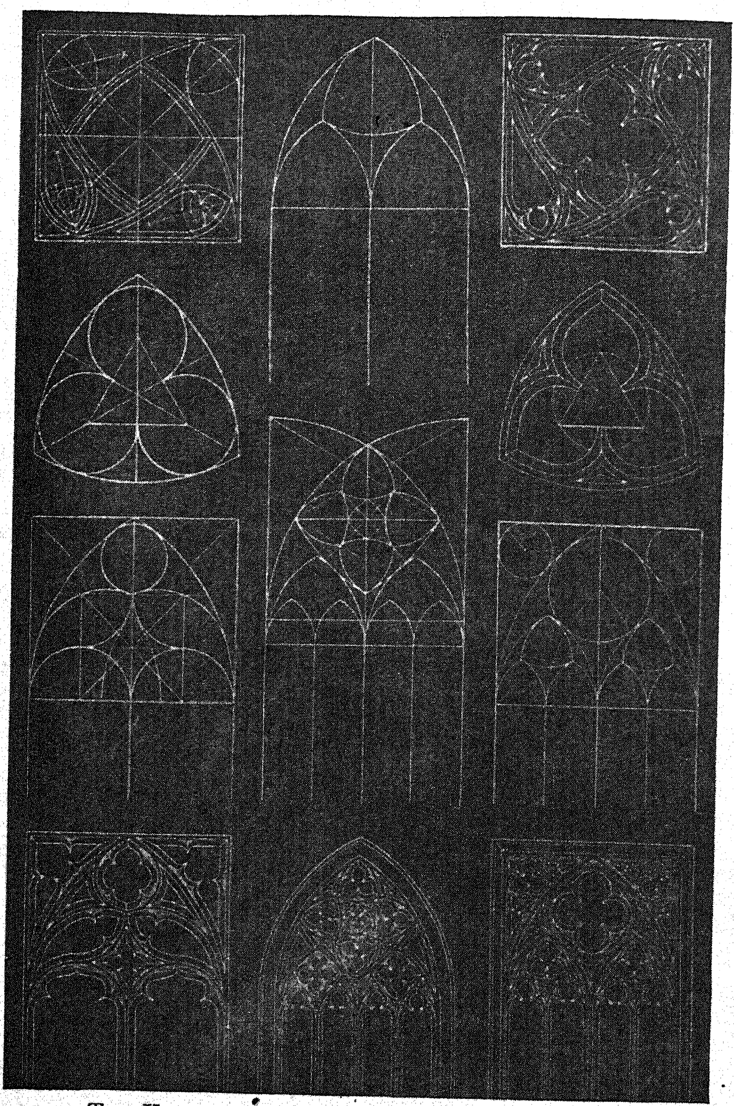
8. The circumference of a circle is equal to the product of twice the radius by the constant number pi.

9. The area of a sector of a circle is equal to the product of the length of its arc by half the radius.

10. The perimeters of two regular polygons having the same number of sides have the same ratio as their apothems.

11. The length of an arc of a circle is $\frac{1}{360}$ times the circumference multiplied by the number of degrees in the arc.

12. The area of an equilateral triangle is equal to one-fourth the square of the side times the square root of three.



THE USE OF THE CIRCLE IN DECORATIVE DESIGN

APPENDIX

REVIEW OF ALGEBRA

Simple Equations

1. Solve for x: $x - (180 - x) = 40$

SOLUTION.
$$x - (180 - x) = 40$$
$$x - 180 + x = 2x - 180 = 40$$
$$x = 110.$$

2. What is the angle which equals five times its complement?

SOLUTION.
Let $x =$ the angle
$90 - x =$ the complement
Then $x = 5(90 - x)$
$$x = 450 - 5x$$
$$6x = 450$$
$$x = 75$$
$$90 - x = 15$$

The angles are 75° and 15°.

3. $x(x + 20) = (x + 6)(x + 6)$

SOLUTION.

$x^2 + 20x = x^2 + 12x + 36$	Multiplying
$20x - 12x = 36$	Transposing
$8x = 36$	Simplifying
$x = \frac{36}{8} = 4\frac{1}{2}$	Dividing by 8

4. $\dfrac{(n - 2)180}{n} = 160$

SOLUTION.

$(n - 2)180 = 160n$	Multiplying by n
$180n - 360 = 160n$	Simplifying
$180n - 160n = 360$	Transposing
$20n = 360$	Simplifying
$n = 18$	Dividing by 20

Square Root

(*Refer to table on page* 454.)

1. Find the square root of 24 correct to the nearest hundredth; to the nearest tenth.

Solution. From the table $\sqrt{24} = 4.899$.

Correct to the nearest hundredth this is 4.90; to the nearest tenth it is 4.9.

2. Find $\sqrt{432}$ correct to the nearest tenth.

Solution. $432 = 144 \times 3$. Hence $\sqrt{432} = \sqrt{144 \times 3} = 12\sqrt{3}$.

From the table $\sqrt{3} = 1.732$; hence $12\sqrt{3} = 20.784$. The answer is 20.8.

3. Find $\sqrt{7470}$ correct to the nearest tenth.

Solution. From the table
$$
\begin{aligned}
86^2 &= 7396 \\
x^2 &= 7470 \\
87^2 &= 7569.
\end{aligned}
$$

The difference between 7569 and 7396 is 173; between 7470 and 7396 is 74.

$$\frac{74}{173} = .43. \quad \text{Hence } \sqrt{7470} = 86.4.$$

Another Method.

$$
\begin{array}{rr}
 & 8\ \ 6.\ 4\ \ 2 \\
 & \overline{74\ 70.00\ 00} \\
 & 64 \\
166 & \overline{10\ 70} \\
 & 9\ 96 \\
1724 & \overline{7400} \\
 & 6896 \\
17282 & \overline{50400} \\
 & \underline{34564}
\end{array}
$$

Hence $\sqrt{7470} = 86.4$.

To reduce a mixed expression to a fraction

1. Change to fractional form: $s^2 - \frac{s^2}{4}$.

Solution. $s^2 - \frac{s^2}{4} = \frac{4s^2 - s^2}{4} = \frac{3s^2}{4}$.

2. Change to fractional form: $c^2 - \left(\frac{b^2 + c^2 - a^2}{2b}\right)^2$.

Solution. $$c^2 - \left(\frac{b^2 + c^2 - a^2}{2b}\right)^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4b^2}$$
$$= \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2}$$

Factoring

Factor the following:

1. $ax + bx + cx$ *Ans.* $x(a + b + c)$.

2. $a^2 - b^2$ *Ans.* $(a - b)(a + b)$.

3. $a^2 - (b^2 - 2bc + c^2)$.

Solution. $$a^2 - (b^2 - 2bc + c^2) = a^2 - (b - c)^2$$
$$= (a - b + c)(a + b - c)$$

4. $4b^2c^2 - (b^2 + c^2 - a^2)^2$

Solution. $$[2bc + (b^2 + c^2 - a^2)][2bc - (b^2 + c^2 - a^2)]$$
$$= (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)$$
$$= [(b + c)^2 - a^2][a^2 - (b - c)^2]$$
$$= (b + c - a)(b + c + a)(a - b + c)(a + b - c)$$

Simultaneous Linear Equations

Solve for x and y:

1. $\begin{cases} 2x + 3y = 70 & (1) \\ 3x + 2y = 80 & (2) \end{cases}$

SOLUTION.

$$\begin{array}{rll} 6x + 9y &= 210 & \text{Multiplying (1) by 3} \\ 6x + 4y &= 160 & \text{Multiplying (2) by 2} \\ \hline 5y &= 50 & \text{Subtracting} \\ y &= 10 & \text{Dividing by 5} \\ 2x + 3 \times 10 &= 70 & \text{Substituting in (1)} \\ 2x = 70 - 30 &= 40 & \text{Transposing and simplifying} \\ x &= 20 & \text{Dividing by 2} \end{array}$$

2. Solve for x, y, and z.

$$\begin{array}{rl} x + y = 5 & (1) \\ x + z = 6 & (2) \\ y + z = 8 & (3) \end{array}$$

Eliminating z by subtracting (3) from (2)

$$x - y = -2 \quad (4)$$

Adding (1) and (4) $x = \frac{3}{2}$

Similarly find y and z.

Quadratic Equations

Solve for x:

1. $(x + 6)(x - 6) = 8.8$

SOLUTION.

$$\begin{aligned} x^2 - 36 &= 8.8 \\ x^2 &= 44.8 \\ x &= 6.7 \text{ to one decimal} \end{aligned}$$

2. $x(x + 8) = 240$

SOLUTION.

$$\begin{aligned} x^2 + 8x &= 240 \\ x^2 + 8x - 240 &= 0 \\ (x + 20)(x - 12) &= 0 \\ x &= -20, +12 \end{aligned}$$

3. $x(16 - x) = 36$

SOLUTION. $16\,x - x^2 = 36$

$$x^2 - 16\,x + 36 = 0$$

To solve by formula, $a = 1,\ b = -16,\ c = 36$

Substitute in $x = \dfrac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}$

$$x = \frac{16 \pm \sqrt{256 - 144}}{2} = \frac{16 \pm \sqrt{112}}{2}$$

$$x = \frac{16 \pm 10.6}{2} = 13.3 \text{ or } 2.7$$

Simplifying Radicals

1. Find the value of $\dfrac{3 - 2\sqrt{5}}{3}$.

SOLUTION. $\dfrac{3 - 2\sqrt{5}}{3} = \dfrac{3 - 2(2.236)}{3} = \dfrac{3 - 4.472}{3} = -.49$

2. Simplify $\sqrt{\dfrac{4\,h^2}{3}}$.

$$\sqrt{\frac{4\,h^2}{3}} = 2\,h\sqrt{\frac{1}{3}} = 2\,h\sqrt{\frac{3}{9}} = \frac{2\,h}{3}\sqrt{3}$$

3. Simplify $\left(\dfrac{3 - \sqrt{5}}{2}\right)^2$.

SOLUTION. $\left(\dfrac{3 - \sqrt{5}}{2}\right)^2 = \dfrac{14 - 6\sqrt{5}}{4} = .146 \text{ or } .15$

FORMULAS USED IN PLANE GEOMETRY

a, b, c, sides of $\triangle ABC$, a being opposite $\angle A$, etc.
(In a right triangle $\angle C = 90°$)
p, perimeter
s, semiperimeter
A, area
r, radius
c, circumference
d, diameter or diagonal
h, altitude; h_a, altitude on side a
b, base
p_b^a, projection of a on b
π, $3\frac{1}{7}$ approx.
a, apothem

(1) $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

(2) h (equilateral triangle) $= \frac{a}{2}\sqrt{3}$

(3) $c^2 = a^2 + b^2 - 2\,bp_b^a$ ($\angle C$ acute)
$c^2 = a^2 + b^2 + 2\,bp_b^a$ ($\angle C$ obtuse)
$c^2 = a^2 + b^2$ ($\angle C$ right)

(4) $d = a\sqrt{2}$ (square)

(5) $c = 2\,\pi r$

(6) Length of an arc $= \dfrac{\text{angle of the arc}}{360} \times 2\,\pi r$

(7) Each angle $= \dfrac{n-2}{n}\,180°$. (regular polygon)

(8) $p : p' = a : a' = b : b' \ldots$ etc. (similar polygons).

(9) $c : c' = r : r' = d : d'$ (circles)

Areas

(10) $A = hb$ (rectangle or parallelogram)

(11) $A = \frac{hb}{2}$ (triangle)

(12) $A = \frac{a^2}{4}\sqrt{3}$ (equilateral triangle)

(13) $A = a^2$ (square)

(14) $A = \sqrt{s(s-a)(s-b)(s-c)}$ (any triangle)

(15) $A = \frac{h}{2}(b_1 + b_2)$ (trapezoid)

(16) $A = \frac{1}{2} d \times d'$ (rhombus)

(17) $A = \pi r^2$ (circle)

(18) $A = \frac{1}{2} ap$ (regular polygon)

(19) $A = \frac{\text{Angle of sector}}{360} \pi r^2$, (sector of circle)

or,

$A = \text{arc of sector} \times \frac{1}{2} r$

(20) A = sector $\pm$ $\triangle$ formed by chord and radii (segment)

(21) $A : A' = a^2 : a'^2$ (similar polygons)

(22) $A : A' = r^2 : r'^2 = d^2 : d'^2$ (circle)

POWERS AND ROOTS

No.	Sqs.	Sq. Roots	Cubes	Cube Roots	No.	Sqs.	Sq. Roots	Cubes	Cube Roots
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	**52**	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	**55**	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	**57**	3,249	7.550	185,193	3.849
8	64	2.828	512	2.000	**58**	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	**65**	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	**68**	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	**69**	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	**70**	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	**71**	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	**73**	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	**77**	5,929	8.775	456,533	4.254
28	784	5.292	21,952	3.037	**78**	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	**81**	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	**82**	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	**83**	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	**84**	7,056	9.165	592,704	4.379
35	1,225	5.916	42,875	3.271	**85**	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	**86**	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	**88**	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	**89**	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	**90**	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	**92**	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.644	804,357	4.531
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	**95**	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	**96**	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	**97**	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.642

AXIOMS

1. *Quantities which are equal to the same quantity, or to equal quantities, are equal to each other.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are multiplied by equals, the products are equal.*

5. *If equals are divided by equals (not zero), the quotients are equal.*

6. *The whole of a quantity is equal to the sum of all of its parts, and is greater than any of its parts.*

7. *A quantity may be substituted for its equal in any expression.*

8. *If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by the same positive number, the results are unequal in the same order.*

9. *If unequals are subtracted from equals, the results are unequal in the opposite order.*

10. *If unequals are added to unequals in the same order, the results are unequal in the same order.*

11. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

12. *Like powers, or like positive roots, of equals are equal.*

POSTULATES AND FUNDAMENTAL THEOREMS

1. *A straight line can be produced to any required length.*

2. *Two straight lines cannot intersect in more than one point.*

3. *Through two given points one and only one straight line can be drawn.*

4. *The length of the line segment connecting two points is the shortest distance between them.*

5. *A circle may be drawn with any point as center and with any line segment as radius.*

6. *All radii of the same circle or of equal circles are equal, and all diameters of the same circle or of equal circles are equal.*

7. *A geometric figure may be moved without changing its size or shape.*

8. *A line segment has one and only one point of bisection.*

9. *All right angles are equal.*

10. *An angle has one and only one bisector.*

11. *Two lines in the same plane must either be parallel or they must intersect.*

12. *Through a given outside point there can be one and only one parallel to a given line.*

13. *A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than the radius.*

14. *A diameter of a circle bisects the circle and the surface inclosed by it; if a line bisects a circle, it is a diameter.*

15. *Two lines perpendicular to intersecting lines must intersect.*

16. *In the same circle, or in equal circles, equal central angles have equal arcs, and conversely.*

17. *In the same circle, or in equal circles, the greater of two unequal central angles has the greater arc, and conversely.*

18. *A central angle has the same measure as its arc.*

19. *Any theorem which has been proved true regarding a regular polygon, and which does not depend on the number of sides of the polygon, is equally true for the circle.*

SYLLABUS OF THE PROPOSITIONS

The section numbers starred are the starred theorems of the College Board. Those in bold face are the ones stressed by the New York Regents.

39. *Equal angles have equal complements.*

40. *Equal angles have equal supplements.*

46. *Vertical angles are equal.*

*64. *If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of another, the triangles are congruent.*

*65. *If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of another, the triangles are congruent.*

***69.** *If a triangle is isosceles, the angles opposite the equal sides are equal.*

71. *An equilateral triangle is equiangular.*

***76.** *If two angles of a triangle are equal, the sides opposite these angles are equal.*

77. *If a triangle is equiangular, it is also equilateral.*

*80. *If the sides of one triangle are equal, respectively, to the sides of another, the triangles are congruent.*

***83.** *Two right triangles are congruent if the hypotenuse and a side of one are equal, respectively, to the hypotenuse and a side of the other.*

87. *If two points are each equidistant from the ends of a segment, they determine the perpendicular bisector of the segment.*

89. a. *Any point on the perpendicular bisector of a segment is equidistant from the ends of the segment.*

b. *Any point equidistant from the ends of a segment is on the perpendicular bisector of the segment.*

91. *An exterior angle of a triangle is greater than either opposite interior angle.*

*97. *One and only one perpendicular can be drawn to a given line through a given point.*

98. *Two right triangles are congruent if the hypotenuse and adjacent angle of one are equal, respectively, to the hypotenuse and adjacent angle of the other.*

100. a. *Any point in the bisector of an angle is equidistant from the sides of the angle.*

b. *Any point equidistant from the sides of an angle is on the bisector of the angle.*

102. 1. *Of two contradictory propositions, one must be true and the other must be false.*

3. *If the conclusion of a correct line of reasoning is shown to be false, then the hypothesis from which the conclusion follows must be false.*

107. *Two lines in the same plane perpendicular to the same line are parallel.*

109. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

***113.** *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*

114. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

115. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

116. *If two angles have their sides respectively parallel, they are either equal or supplementary.*

118. *If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.*

119. *If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.*

120. *If two lines are cut by a transversal so that two*

interior angles on the same side of the transversal are supplementary, the lines are parallel.

121. *Two lines parallel to a third line are parallel to each other.*

***123.** *The sum of the angles of a triangle is equal to a straight angle.*

124. *A triangle can have but one right angle or one obtuse angle.*

125. *The acute angles of a right triangle are complementary.*

126. *If two angles of one triangle are equal, respectively, to two angles of another triangle, the third angles are equal.*

127. *If two triangles have a side, an adjacent angle, and the opposite angle of the one equal, respectively, to the corresponding parts of the other, the triangles are congruent.*

128. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

129. *If two angles have their sides, respectively, perpendicular, they are either equal or supplementary.*

133. *The sum of the angles of a polygon of n sides is $(n-2)$ straight angles.*

134. *The sum of the exterior angles of a polygon made by producing each of the sides in succession is two straight angles.*

141. *The opposite sides of a parallelogram are equal and the opposite angles are equal.*

142. *A parallelogram is divided into two congruent triangles by either diagonal.*

143. *Segments of parallel lines cut off by parallel lines are equal.*

145. *Two parallel lines are everywhere equidistant.*

146. *If the opposite angles of a quadrilateral are equal, it is a parallelogram.*

147. *The diagonals of a parallelogram bisect each other.*

148. *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

***149.** *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

***150.** *If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.*

***152.** *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*

153. *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.*

154. *If a line connects the mid-points of two sides of a triangle, it is parallel to the third side.*

155. *A line segment connecting the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*

157. *The median of a trapezoid is parallel to the bases and equal to half their sum.*

159. *In a right triangle the median to the hypotenuse is equal to half the hypotenuse.*

160. *In a 30°–60° right triangle, the hypotenuse is double the side opposite the 30° angle.*

161. *If the hypotenuse of a right triangle is double one of the sides, then the acute angle opposite that side is 30°, while the other one is 60°.*

170. *If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.*

172. *If two angles of a triangle are unequal, the sides*

opposite these angles are unequal, and the side opposite the greater angle is the greater.

173. *The perpendicular is the shortest segment that can be drawn from a given point to a given line.*

174. *The shortest segment that can be drawn from a given point to a given line is the perpendicular from the point to the line.*

177. *If two sides of one triangle are equal, respectively, to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

178. *If two sides of one triangle are equal, respectively, to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*

*184. *Through any three given points not in a straight line one circle, and only one, can be drawn.*

185. *A straight line or a circle cannot intersect a circle in more than two points.*

191. *In the same circle, or in equal circles, if two arcs are equal, their chords are equal.*

193. *In the same circle, or in equal circles, if two chords are equal, their arcs are equal.*

***195.** *A diameter perpendicular to a chord bisects the chord and its arcs.*

196. *A diameter which bisects a chord (not a diameter) is perpendicular to the chord.*

197. *The perpendicular bisector of a chord passes through the center of the circle.*

200. *If two circles intersect, the line of centers is the perpendicular bisector of their common chord.*

***202.** *In the same circle, or in equal circles, chords equidistant from the center are equal.*

***203.** *In the same circle, or in equal circles, equal chords are equidistant from the center.*

205. *In the same circle, or in equal circles, if two minor arcs are unequal, the greater arc has the greater chord.*

206. *In the same circle, or in equal circles, if two chords are unequal, the greater chord has the greater minor arc.*

208. *In the same circle, or in equal circles, if two chords are unequal, the greater chord is nearer the center.*

209. *In the same circle, or in equal circles, if two chords are unequally distant from the center, the one nearer the center is the greater.*

214. *A line perpendicular to a radius at its outer extremity is tangent to the circle.*

215. *The tangent to a circle at a given point is perpendicular to the radius drawn to that point.*

216. *A line perpendicular to a tangent at the point of contact passes through the center of the circle.*

217. *A line from the center of a circle, perpendicular to a tangent, passes through the point of contact.*

220. *Two tangents to a circle from an outside point are equal and make equal angles with the line joining that point to the center.*

222. *If two circles are tangent, the line of centers passes through the point of contact.*

226. *Two parallel lines intercept equal arcs on a circle.*

***241.** *An inscribed angle has the same measure as half of its intercepted arc.*

242. *Angles inscribed in the same arc or in equal arcs are equal.*

243. *An angle inscribed in a semicircle is a right angle.*

245. *An angle formed by a tangent and a chord from the point of contact has the same measure as half its intercepted arc.*

246. *An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.*

248. *An angle formed by two secants, by a secant and a tangent, or by two tangents intersecting outside the circle has the same measure as half the difference between the intercepted arcs.*

252. *The locus of points at a given distance from a given point is a circle with the given point as center and with the given distance as radius.*

***254.** *The locus of points equidistant from two given points is the perpendicular bisector of the segment connecting the points.*

***256.** *The locus of points equidistant from the sides of an angle is the bisector of the angle.*

257. *The locus of points equidistant from two intersecting straight lines is the pair of lines bisecting the angles formed.*

258. *The locus of points equidistant from two parallel lines is the line parallel to each of them and midway between them.*

259. *The locus of points at a given distance from a given line is a pair of lines, one on either side of the given line, each parallel to the given line, and at the given distance from it.*

260. *The locus of the vertex of the right angle of a right triangle having a given hypotenuse is a circle having the given hypotenuse as diameter.*

270. *The perpendicular bisectors of the sides of a triangle are concurrent.*

273. *The bisectors of the angles of a triangle are concurrent.*

276. *The altitudes of a triangle are concurrent.*

278. *The medians of a triangle intersect in a point which is two-thirds of the distance from any vertex to the mid-point of the opposite side.*

281. *The locus of the vertex C of a triangle with a given base AB and given angle C is the arc of a circle in which angle C can be inscribed which arc has AB as a chord.*

289. 1. *In any proportion, the product of the means equals the product of the extremes.*

2. *In any proportion, the first term is to the third as the second term is to the fourth.*

3. *In any proportion, the second term is to the first as the fourth is to the third.*

4. *If the two numerators of a proportion are equal, the denominators are equal.*

5. *If three terms of one proportion are equal, respectively, to three terms of another proportion, the remaining terms are equal.*

6. *If the product of two numbers is equal to the product of two other numbers, either two may be made the means in a proportion in which the other two are the extremes.*

7. *In a proportion, the sum of the first two terms is to the second (first) as the sum of the last two is to the fourth (third).*

8. *In a proportion, the difference between the first two terms is to the second (first) as the difference of the last two is to the fourth (third).*

293. *A line parallel to one side of a triangle and intersecting the other two sides divides those two sides proportionally.*

294. *One side is to either of its segments as the other side is to the corresponding segment.*

298. *The segments cut off on two transversals by a series of parallels are proportional.*

***300.** *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

***305.** *Two triangles are similar if two angles of one are equal, respectively, to two angles of the other.*

306. *Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.*

***312.** *Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.*

***314.** *Two triangles are similar if their corresponding sides are proportional.*

317. *The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*

319. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*

321. A. *The bisector of the interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side of the triangle harmonically.*

322. *In any right triangle, the perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two triangles similar to the given triangle.*

323. I. *The two triangles are similar to each other.*

II. *The perpendicular is the mean proportional between the segments of the hypotenuse.*

III. *Either side is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to it.*

324. *The perpendicular to the diameter of a circle from any point on the circle (a) is the mean proportional between the segments of the diameter; and*

(b) the chord from that point to either extremity of the diameter is the mean proportional between the diameter and the segment of the diameter adjacent to that chord.

***326.** *In any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.*

328. A. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of those sides by the projection of the other side on it.*

329. A. *In any obtuse triangle, the square of the side opposite an obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of those sides by the projection of the other side on it.*

***333.** *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*

335. *If, from a point outside a circle, a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.*

336. *If, from an external point, secants are drawn to a circle, the product of each secant by its external segment is a constant.*

341. *The area of a rectangle is equal to the product of its base and altitude.*

***343.** *The area of a parallelogram is equal to the product of its base by its altitude.*

344. *Parallelograms having equal bases and equal altitudes are equal in area.*

345. *Two parallelograms are to each other as the products of their bases and altitudes.*

346. *Parallelograms having equal altitudes are to each other as their bases.*

347. *Parallelograms having equal bases are to each other as their altitudes.*

348. *The area of a triangle is equal to half the product of its base by its altitude.*

349. *Two triangles having equal bases and equal altitudes are equal in area.*

350. *Two triangles are to each other as the products of their bases and altitudes.*

351. *Triangles having equal altitudes are to each other as their bases.*

352. *Triangles having equal bases are to each other as their altitudes.*

***353.** *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*

354. *The area of a trapezoid is equal to the product of its altitude and the segment connecting the mid-points of the legs.*

356. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*

360. *In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

364. *In a series of equal ratios, the sum of the numerators is to the sum of the denominators as any numerator is to its denominator.*

365. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*

*367. *If two polygons are similar, they can be divided into triangles which are similar and similarly placed.*

368. *If two polygons can be divided into triangles which are similar and similarly placed, the polygons are similar.*

***372.** *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*

***373.** *The areas of two similar polygons have the same ratio as the squares of any two corresponding sides.*

374. A. *In any right triangle, a polygon constructed on the hypotenuse is equal in area to the sum of similar polygons constructed on the other two sides.*

381. *If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon.*

382. *An equilateral polygon inscribed in a circle is a regular polygon.*

383. *If lines are drawn from the mid-point of each arc determined by a side of a regular inscribed polygon, to its extremities, a regular inscribed polygon of double the number of sides is formed.*

384. *Regular inscribed polygons of 4, 8, 16, 32, etc., sides can be constructed.*

385. *Regular inscribed polygons of 3, 6, 12, 24, 48, etc., sides can be constructed, and, by joining the alternate vertices of an inscribed hexagon, an inscribed equilateral triangle is formed.*

387. *If a circle is divided into equal arcs, the tangents at the points of division form a regular circumscribed polygon.*

388. *If tangents are drawn at the mid-points of the arcs of adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.*

***390.** *A circle can be circumscribed about any regular polygon.*

391. *A circle can be inscribed in any regular polygon.*

392. *Each central angle of a regular polygon of n sides is* $\frac{360°}{n}$.

393. *Each angle of a regular polygon of n sides is* $\frac{n-2}{n}$ *straight angles.*

394. *Since the sides of a regular polygon are equal, if each side is s, and there are n sides, the perimeter of a regular polygon is ns.*

***395.** *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*

397. *Regular polygons of the same number of sides are similar.*

399. *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.*

400. *The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.*

404. A. *Regular polygons of 5, 10, 20, 40, etc., sides can be inscribed in a circle.*

406. A. *Regular polygons of 15, 30, 60, 120, etc., sides can be inscribed in a circle.*

411. *The circumferences of two circles have the same ratio as their radii.*

412. *The circumferences of two circles have the same ratio as their diameters.*

413. *The ratio of the circumference to the diameter of a circle is a constant: that is, it is the same for any two circles.*

415. *The circumference of a circle is equal to the product of its radius by twice the constant number* π, *or,* $C = 2\pi r$.

416. *The length of an arc of a circle in linear units has the same ratio to the circumference as the number of degrees in the arc has to 360°.*

417. *The area of a circle is equal to half the product of its circumference by its radius.*

418. *The area of a circle is equal to the product of the constant number* π *by the square of the radius.*

420. *The area of a sector has the same ratio to the area of the circle as the angle of the sector has to 360°.*

421. *The area of a sector is equal to the product of the length of its arc by half the radius.*

422. *The areas of two circles have the same ratio as the squares of their radii, or as the squares of their diameters.*

INDEX

(Numbers refer to pages)

Abbreviations, xiv.
Abûl-Wefa, 296.
Acute, angle, 22; triangle, 23.
Adjacent angles, 21.
Ahmes, 21, 365, 425, 427.
Algebra Review, 447.
Alternate, exterior, and interior angles, 119.
Altitude, of a parallelogram, 136; of a trapezoid, 136; of a triangle, 74.
Analysis, 86.
Angle, acute, 22; bisected, 25; bisector of, 304; central, 175, 222; degree, 221; of depression, 299; of elevation, 299; exterior, 93; initial side, 19, 122; inscribed, 224; measuring of, 27; mirror, obtuse, 22; reflex, 22; right, 22; sides, 18; size of, 18, 154; straight, 22; terminal side, 19, 122; vertex, 18.
Angles, adjacent, 21; alternate, 119; complementary, 32; corresponding, 119; equal, 19; exterior, 119; interior, 119; supplementary, 32; vertical, 36.
Antecedents, 280.
Appendix, 447.
Apothem, 406.
Arc, 7; degree, 221; major, 175; minor, 175; minute, 221; negative, 234; positive, 234; second, 221.
Arcs, complementary, 222; supplementary, 222.
Archimedes, 16, 66, 427, 437.
Area, of a circle, 424; of a parallelogram, 346; of a rectangle, 343; of a surface, 343; of a trapezoid, 350; of a triangle, 348.
Arithmetic in Nine Sections, 283.
Aryabhatta, 427.
Auxiliary lines, **72.**
Axiom, 39.
Axioms, list of, 455.

Babylonians, 427.
Base, of an isosceles triangle, 10; of a parallelogram, 136.
Bases, of a trapezoid, 136.
Bhaskara, 361.
Bisector, of an angle, 25; of a segment, 15; perpendicular, 30.
Broken line, 6.

Center, of a circle, 6, 173; of gravity, 254; of a regular polygon, 406.
Central angle, 175; of a regular polygon, 406.
Centroid, 254.
Chord, 7, 173.
Circle, 6, 173; arc, 7; area, 420; center, 6, 173; central angle, 175; chord, 7, 173; circumference, 420; circumscribed, 182; diameter, 7, 173; escribed, 252; inscribed, 195; radius, 7, 173; sector, 424; segment, 423.
Circles, concentric, 177; line of centers, 182; tangent, externally, internally, 197.
Circumcenter, 250.
Circumference, 420, 421.
Circumscribed, circle, 182; polygon, 195.
Clinometer, 333.
Commensurable, 284.
Common tangent, 198.
Compasses, 11; proportional, 331.
Complement, 32.
Concave polygon, 132.
Concentric circles, 177.
Conclusion, 71.
Concurrent, 249.

Congruent, 21.
Congruent figures, 45.
Conic sections, 243.
Consequents, 280.
Converse, 75.
Convex polygon, 132.
Corollary, 73.
Corresponding, angles, 119; angles of polygons, 290; parts, 49; sides of polygons, 290.
Cosine, 295; table of, ratios, 298.
Cross-staff, 332.
Curved line, 6.
Cycloid, 245.

Decagon, 132.
Definition, properties of, 4.
Degree, angle, 27, 221; arc, 221.
Demonstration, 67.
Depression, angle, 299.
Diagonal, of a polygon, 132.
Diameter, 7, 173.
Direct proof, 100.
Distance, between parallel lines, 137; from a point to a line, 98, 152.
Divine proportion, 413.
Division, external, 307; harmonic, 308; internal, 307.
Drumheads, 332.
Duplicating the cube, 358.

Egyptians, 2, 19.
Elevation, angle, 299.
Ellipse, 243.
Equiangular, 23.
Equilateral, 8, 10.
Escribed circle, 252.
Euclid, 66, 102, 311, 343, 413.
Excenter, 252.
Exterior angles, 93, 119.
External division, 307.
External tangent, 198.
Extreme and mean ratio, 413.
Extremes, 278.

Figures, congruent, 45.
Formulas, 380, 452.
Fourth proportional, 282.
Fundamental theorems, 455.

Galileo, 335.
Gauss, 420.
Geometric square, 333.
Golden section, 413.
Greeks, 2, 3, 6, 16, 27, 102, 187, 283, 294, 296, 310, 313, 343, 354, 358, 362, 383, 437.

Harmonic division, 308.
Harpedonaptae, 19.
Hebrews, 427.
Heptagon, 132.
Hero, 354.
Hexagon, 8, 132.
Hipparchus, 297.
Hippocrates, 187, 437.
Historical notes, 2, 3, 6, 16, 19, 21, 27, 77, 98, 102, 187, 283, 294, 296, 297, 310, 311, 313, 320, 335, 343, 354, 358, 361, 362, 365, 383, 407, 413, 420, 425, 427, 437.
Hyperbola, 243.
Hypotenuse, 83.
Hypothesis, 71.
Hypsometer, 333.

Incenter, 252.
Indirect method, 101.
Inequalities, 150, 186.
Inscribed, angle, 224; circle, 195; polygon, 182.
Intercepted, 142, 224.
Interior angles, 119.
Interpolation, 296.
Internal, division, 307; tangent, 198.
Isosceles, trapezoid, 145; triangle, 10.

K'iu-chang Suan-shu, 283.

Legendre, 66.
Legs, of an isosceles triangle, 10; of a right triangle, 83.
Lincoln, 77.
Line, broken, 6; curved, 6; straight, 6; of centers, 182.
Lines, auxiliary, 72; concurrent, 249; parallel, 104; perpendicular, 22; transversal, 119.
Line segment, 6; bisected, 15.
Linkages, 244, 320, 358.
Locus, 235, 238, 241, 246.

Major arc, 175.
Maps and plans, 303.
Mariner's Compass, 29.
Mean proportional, 282.
Means, 278.
Measure, of an angle, 27; of an arc, 221; of a central angle, 222; of an inscribed angle, 224; numerical, 221.
Median, of a trapezoid, 145; of a triangle, 16, 74.
Mid-point of a segment, 15.
Minor arc, 175.
Minutes, 28.
Mirror, angle, 22.
Mutually equiangular, 290.

Negative arc, 234.
Nonagon, 132.

Obtuse, angle, 22; triangle, 23.
Octagon, 132.
Optical illusion, 35.
Orthocenter, 253.

Pantograph, 334.
Pappus, 362.
Parabola, 243.
Parallel lines, 104; distance between, 137.
Parallelogram, 136; base of, 136; altitude of, 136.
Peaucellier, 320.
Pentagon, 132.
Perimeter, 198.
Perpendicular bisector, 30.
Perpendicular lines, 22.
Pi (π), 422; computation of, 427.
Plane, 1.
Plane table, 372.
Plans and maps, 303.
Plato, 2, 66.
Plutarch, 98.
Point, of contact, 192; of tangency, 192.
Polygon, 9; circumscribed, 195; concave, 132; convex, 132; diagonal of, 132; inscribed, 182; perimeter of, 198; regular, 133, 397; sum of the angles of, 132; vertices, 132.
Polygons, classified, 132; mutually equiangular, 290; similar, 290.
Positive arc, 234.
Postulate, 13.
Postulates, list of, 455.
Powers and roots, table of, 454.
Practical Applications of Geometry, 60-62, 131-132, 134-135, 141-142, 164-166, 212-214, 268-270, 294, 330-336, 359, 438-439.
"Problems of Antiquity," 358.
Proclus, 313.
Projection, 313; negative and positive, 317; of a point, 313; of a segment, 314.
Proof, 36, 67, 100; by analysis, 84; indirect proof, 101; reductio ad absurdum, 107.
Proportion, 277; terms, 278.
Proportional, compasses, 331; fourth, mean, third, 282.
Proposition, 67.
Protractor, 28.
Ptolemy, 27.
Pythagoras, 66, 187, 310, 313, 413.
Pythagorean theorem, 310, 311, 361.

Quadrant, 175.
Quadrilateral, 132.
Quatrefoil, 432.

Radius, of a circle, 7, 173; of a regular polygon, 406.
Ratio, 277; extreme and mean, 413; of similitude, 290; trigonometric, 295, 298.
Rectangle, 136.
Reductio ad absurdum, 107.
Reflex angle, 22.
Regular polygon, 133, 397; apothem, 406; center, 406; central angle, 406; perimeter, 406.
Resultant, 141.
Review of Algebra, 447.
Reviews, 57-60, 109-112, 161-166, 209-214, 263-270, 324-336, 382-389, 428-440.
Rhombus, 136.
Riemann, 66.
Right angle, 22.
Right triangle, 23; hypotenuse and legs of, 83; 30°-60°, 147.
Rope Stretchers, 19.

Scale, 303.
Scalene triangle, 10.
Secant, 191, 318; external and whole segment, 318.
Seconds, 28.
Sector, 424.
Segment, of a circle, 423; of a line, 6; divided, externally, internally, 307; divided harmonically, 308; projection of, 314.
Semicircle, 175.
Sextant, 28.
Similar, figures, 277; polygons, 290.
Sine, 295; table of, ratios, 298.
Size of an angle, 18, 154.
Socrates, 3.
Sophists, 187.
Square, 136; geometric, 333.
Square root, 448; table of, 454.
Squaring the circle, 358.
Straight, angle, 22; line, 6.
Straightedge, 6.
Study Aids, 70.
Summaries 55-56, 107-108, 158-160, 208-209, 260-262, 323-324, 380-381, 427-428.
Superposition, 45.
Supplement, 32.
Symmetry, 9.

Table, Powers and Roots, 454; Trigonometric Ratios, 298.
Tangent, common external and internal, 198; historical note, 296; of an angle, 295; to a circle, 318; table of, ratios, 298.
Tangent circles, 197.
Telemeter, 334.
Terminal side of an angle, 19, 122.
Tests, 62-66, 112-115, 115-117, 166-169, 169-171, 214-216, 217-219, 270-273, 273-276, 336-339, 339-341, 389-392, 392-395, 440-442, 442-445.
Thales, 6, 66, 294, 296.
Theorem, 34, 67; parts of, 70.
Third proportional, 282.
Three famous problems of antiquity, 358.
Transit, 28, 30.
Transversal, 119.
Trapezoid, 136; isosceles, 145; median of, 145.
Trefoil, 334.
Triangle, 10; acute, 23; altitude of, 74; congruent, 21, 45; equiangular, 23; equilateral, 10; isosceles, 10; median of, 16, 74; obtuse, 23; right, 23; scalene, 10.
Triangles, mutually equiangular, 291; similar, 21.
Trigonometric ratio, 295; table of, 298.
Trisecting any angle, 358.

Unit of measure, of an angle, 27, 221; of an arc, 221; of a surface, 343.

Vertex, of an angle, 18; of a polygon, 9; of a triangle, 10.
Vertical angles, 36.
Vertices of a polygon, 9, 132.